

Curvature perturbation and waterfall dynamics in hybrid inflation

Ali Akbar Abolhasani^{1,3,*}, Hassan Firouzjahi^{2,†} and Misao Sasaki^{3,4‡}

¹*Department of Physics, Sharif University of Technology, Tehran, Iran*

²*School of Physics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5531, Tehran, Iran*

³*Yukawa Institute for theoretical Physics, Kyoto University, Kyoto 606-8502, Japan and*

⁴*Korea Institute for Advanced Study, 207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722, Republic of Korea*

(Dated: October 13, 2011)

We investigate the parameter spaces of hybrid inflation model with special attention paid to the dynamics of waterfall field and curvature perturbations induced from its quantum fluctuations. Depending on the inflaton field value at the time of phase transition and the sharpness of the phase transition inflation can have multiple extended stages. We find that for models with mild phase transition the induced curvature perturbation from the waterfall field is too large to satisfy the COBE normalization. We investigate the model parameter space where the curvature perturbations from the waterfall quantum fluctuations vary between the results of standard hybrid inflation and the results obtained here.

I. INTRODUCTION

Thanks to vast data obtained from recent cosmological observations such as WMAP7 [1] inflation [2] has emerged as the leading theory of the early universe and structure formation. The simplest class of models of inflation is based on a scalar field, the inflaton field, coupled minimally to gravity [3, 4]. The inflaton field potential is flat enough to support long enough period of inflation to solve the horizon and flatness problems associated with the big bang cosmology. These simple models predict almost scale invariant, almost Gaussian and almost adiabatic perturbations which are very well consistent with the Cosmic Microwave Background (CMB) observations [1]. However, with the help of ever increasing accurate cosmic measurements different inflationary models can be discriminated based on their predictions for the power spectrum spectral index or the amount of gravitational waves.

Recently there has been a revival of interest in hybrid inflation [5, 6], specifically the dynamics of waterfall field and whether or not significant large scale curvature perturbations can be produced during the waterfall phase transition [7–12], for earlier works see also [13–16]. In this work we search the parameter space of hybrid inflation in details and study the effects of the waterfall field dynamics carefully. We show that depending on the model parameters inflation can take place at different stages [17–19]. In conventional hybrid inflation [5, 6] inflation ends abruptly by the waterfall transition. Here we find situations in which inflation can proceed as in chaotic inflation [4] even after the waterfall transition. This is a new model of double inflation [20] in the context of hybrid inflation. Interestingly, as we shall see, in some situations inflation can actually have three extended stages. The key parameter is the ratio ϕ_c/M_P where ϕ_c is the value of the inflaton field at the time of waterfall. With $\phi_c/M_P \gg 1$ one can get a second period of inflation even after a sharp waterfall phase transition.

We investigate the curvature perturbations in these models. The key constraint is that at the background the waterfall field ψ is turned off so one has to introduce an effective trajectory $\sqrt{\langle \delta\psi^2 \rangle}$ from the quantum fluctuations of the waterfall field. This in turn plays crucial roles when one calculates the curvature perturbation power spectrum. We exclude a large class of model parameters where the curvature perturbations cannot be normalized to the COBE/WMAP amplitude [21, 22].

The rest of paper is organized as follows. In section II we present our setup. In sections III, IV and V we consider a particular model in which inflation has three extended stages. After presenting the background solutions, we study the waterfall quantum fluctuations and the induced curvature perturbations in this model using the δN -formalism. In section VI we investigate other limits of the model parameter space and calculate the curvature perturbations. We investigate the model parameters where the induced curvature perturbations from the waterfall float between the results obtained in sections III-V and the results of standard hybrid inflation. Some technical calculations of the waterfall field dynamics are relegated into Appendices A, B and C.

*Electronic address: abolhasani-AT-mail.ipm.ir

†Electronic address: firouz-AT-mail.ipm.ir

‡Electronic address: misao-AT-yukawa.kyoto-u.ac.jp

II. HYBRID INFLATION AND ITS PARAMETERS SPACE

As mentioned above we would like to investigate different limits of hybrid inflation potential parameter space and examine the effects of the waterfall transition on curvature perturbations. The potential has the form

$$V(\phi, \psi) = \frac{\lambda}{4} \left(\psi^2 - \frac{M^2}{\lambda} \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2, \quad (2.1)$$

where ϕ is the inflaton field, ψ is the waterfall field, and λ and g are two dimensionless couplings. The potential has global minima at $(\phi, \psi) = (0, \pm M/\sqrt{\lambda})$.

In our analysis below, we retain some key properties of standard hybrid inflation as proposed in [5, 6] while relaxing other conditions. More specifically, as in conventional model of hybrid inflation, we keep the assumption of the vacuum domination. That is, during the first stage of inflation and before the phase transition starts, inflation is driven basically by the vacuum energy $M^4/4\lambda$. Secondly, we assume that the waterfall field is very heavy so it quickly relaxes to its instantaneous minimum $\psi = 0$ before the phase transition. Besides these two key assumptions, we relax the other conditions on model parameters. For example, in the original version of hybrid inflation ϕ is sub-Planckian, while here we relax this condition and shall assume ϕ_c/M_P to be arbitrary where $\phi_c \equiv M/g$ is the value of the inflaton field at the onset of waterfall. Furthermore, we keep the ratio g^2/λ arbitrary whereas in conventional models of hybrid inflation it is usually assumed that $g^2 \sim \lambda$. Before we proceed we mention that similar studies have been performed in [23, 24]. These works mainly concentrated on the classical fields dynamics whereas in our analysis the quantum fluctuations of the waterfall field play key roles.

The first period of inflation takes place at $\phi > \phi_c$ and $\psi = 0$. As stressed above, as in a conventional model of hybrid inflation we assume this stage is vacuum-energy dominated,

$$\frac{M^4}{4\lambda} \gg \frac{1}{2} m^2 \phi^2, \quad (2.2)$$

with the Hubble parameter given by

$$H^2 = \frac{V(\phi, 0)}{3M_P^2} \simeq \frac{M^4}{12\lambda M_P^2}, \quad (2.3)$$

where $M_P = 1/\sqrt{8\pi G}$ is the (reduced) Planck mass. We assume $m^2 \ll H^2$ so that ϕ is sufficiently slow-rolling during the first period of inflation before the waterfall. Also, as mentioned before, we also assume that the waterfall field is very heavy so that $\psi = 0$ is classically realized before the waterfall phase transition. These two assumptions about the masses of the inflaton and the waterfall fields can be translated into dimensionless parameter α and β defined by

$$\alpha \equiv \frac{m^2}{H^2}, \quad \beta \equiv \frac{M^2}{H^2}. \quad (2.4)$$

The fact that ϕ is slow-rolling implies that $\alpha \ll 1$ whereas the assumption that ψ is very heavy requires $\beta \gg 1$.

The condition of vacuum domination, for ϕ near ϕ_c , can be written as

$$\frac{\phi_c^2}{M_P^2} \ll \frac{6}{\alpha}. \quad (2.5)$$

Knowing that $\alpha \ll 1$ this constraint indicates that there is no sub-Planckian restriction. Indeed one can also assume $\phi_c \gg M_P$ and a second stage of inflation may follow after a sharp phase transition. Alternatively, one can cast the vacuum domination condition in the form

$$C \equiv \frac{\beta g^2}{\alpha \lambda} > 1. \quad (2.6)$$

This condition indicates a hierarchy of the model parameters required to have the vacuum domination. However, the vacuum domination condition expressed by Eq. (2.5) shed more lights because it singles out the ratio ϕ_c/M_P as the key parameter which effectively distinguishes different limits of hybrid inflation as we shall see below.

After $\phi < \phi_c$ the waterfall becomes tachyonic triggering an instability in the system. We assume that this phase transition is mild enough such that it will take a few tens of e -folds for the waterfall field to settle down to its global minimum. Again this is possible if ϕ stays near the critical point for a sufficiently long time. As we shall see below this condition requires $m^2 M^2 \ll H^4$. After the waterfall transition the quantum fluctuations of ψ becomes

tachyonic and will grow exponentially. The accumulative effects of these growing modes, i.e. $\langle \delta\psi^2 \rangle$, through the interaction term changes the dynamics of the system. Soon after the gradient term in the waterfall field dynamics becomes negligible compared to the effective mass term the quantum fluctuations $\delta\psi$ obey the same equation as the background homogeneous ψ equation. Hence they effectively behave like a homogeneous classical background, $\psi_{\text{classical}} = \sqrt{\langle \delta\psi^2 \rangle}$, however see also [11].

The tachyonic growth of $\delta\psi$ continues until the self interaction term $\lambda\psi^4$ becomes important. We call this stage the waterfall stage. It ends when ψ settles to its global minimum $\psi_{\min} = \frac{\pm M}{\sqrt{\lambda}}$ and the symmetry breaking is completed. The other important effect of the tachyonic growth of $\delta\psi$ is its back-reaction on the inflaton field via the interaction $g^2\phi^2\psi^2$. This can increase the effective mass of the inflation field significantly ending its slow-roll sharply. The key question is which of the two back-reactions, $\lambda\psi^4$ or $g^2\phi^2\psi^2$, becomes important sooner. In standard hybrid inflation with β extremely large, say $\beta \sim 10^3$, $\phi_c \ll M_P$ and $g^2 \sim \lambda$ these two back-reactions become important more or less at the same time. In this view the time of end of inflation is nearly the same as the time when the symmetry breaking is complete.

In this work we distinguish the completion of symmetry breaking and the termination of inflation as two different phenomena. We will show that, depending on the model parameters, one can consider situation where the waterfall stage and symmetry breaking completion happens at an early stage of inflation, followed by a long period of the second stage of inflation. As we shall see the key parameter in the analysis is the ratio ϕ_c/M_P .

The effective mass of the inflaton can be read as

$$m_{eff}^2 = m^2 + g^2 \langle \delta\psi^2 \rangle \quad (2.7)$$

Knowing that $m^2 \ll H^2$ (or equivalently $\alpha \ll 1$), inflation can only end when $g^2 \langle \delta\psi^2 \rangle \simeq H^2 \simeq M^4/12\lambda M_P^2$. Denoting this value of ψ by ψ_{trig} which triggers the fast roll of ϕ towards its minimum, we find

$$\psi_{trig}^2 \simeq \frac{M^4}{12\lambda g^2 M_P^2} = \frac{\phi_c^2}{12M_P^2} \psi_{\min}^2, \quad (2.8)$$

where $\psi_{\min} = \frac{\pm M}{\sqrt{\lambda}}$. For $\phi_c/M_P < \sqrt{12}$, one has $\psi_{trig} < \psi_{\min}$ so the back-reaction of ψ on ϕ becomes important sooner than the self-interaction $\lambda\psi^4$ so inflaton field rolls quickly to its minimum before the symmetry breaking completion. During this transition, inflation may proceed briefly via the remaining vacuum energy but inflation will end quickly once the waterfall field nearly settles down to its minimum. This is the case in conventional hybrid inflation. On the other hand, for $\phi_c/M_P > \sqrt{12}$ one has $\psi_{trig} > \psi_{\min}$, meaning that the self interaction $\lambda\psi^4$ becomes more important than the back-reaction $g^2\phi^2\psi^2$. As a result, the waterfall field settles down to its minimum while inflation still proceeds afterwards as in chaotic inflation. Of course the dynamics of symmetry breaking is a violent phenomena which can cause a sudden violation of the slow-roll conditions. However, due to the attractor nature of the chaotic inflation potential, the slow-roll conditions are restored quickly for the second period of inflation.

In our analysis it is convenient to use the number of e -folds $dN = Hdt$ as a clock. For convenience we set $N = 0$ at the time when the comoving scale corresponding to the present Hubble radius crossed the horizon during inflation. We denote the number of e -folds till the time when the waterfall field becomes unstable at $\phi = \phi_c$ by $N = N_c$. Taking this epoch as the reference point we also define $n \equiv N - N_c$, so for the period before phase transition (after phase transition) one has $n < 0$ ($n > 0$).

A. $\phi_c^2 > 12M_P^2$

For $\phi_c^2 > 12M_P^2$, one has $\psi_{trig} > \psi_{\min}$ and the back-reaction of ψ on itself becomes important first. The time when it takes for the waterfall field to settle to its global minimum depends on the sharpness of the phase transition. Denote by $n = n_p$ the number of e -folds it take for the waterfall field to settle down to its minimum. For a very sharp phase transition, $n_p \lesssim 1$ whereas for a mild phase transition $n_p \gg 1$.

For $n > n_p$, the effective potential for inflaton field is $V_{eff} = \frac{1}{2}m_{eff}^2\phi^2$ with m_{eff} given by Eq. (2.7). For $\phi_c^2 > 12M_P^2$ this leads to a period of chaotic inflation where ϕ evolves as

$$\phi^2(n) = \phi_p^2 - 4M_P^2(n - n_p), \quad (2.9)$$

where ϕ_p indicates the value of ϕ at the time of n_p . As in chaotic inflation, inflation ends at $\phi = \phi_f$ where the slow-roll conditions are violated. The number of e -folds during the chaotic era, Δn_{ch} , is

$$\Delta n_{ch} = \frac{\phi_p^2 - \phi_f^2}{4M_P^2} \quad (2.10)$$

The total number of e -folds, therefore, is

$$N_{tot} = N_c + n_p + \Delta n_{ch} . \quad (2.11)$$

Here we have called the number of e -folds from $N = 0$ until the end of inflation the total number of e -folds for convenience, but it should be noted that this number is nothing to do with the actual total number of e -folds of inflation which may well be almost infinite.

To solve the flatness and the horizon problem we require $N_{tot} \gtrsim 60$. This picture only depends on the assumption that $\phi_c^2 > 12M_P^2$ and is independent of the values of β and the ratio of the couplings g^2/λ . As we shall see below, the sharpness of the phase transition (the magnitude of n_p) depends on the parameter $\kappa_\psi \simeq \alpha\beta/3$. If $\kappa_\psi \ll 1$ then phase transition is mild and $n_p > 1$. In this limit, as can be seen from Eq. (2.11), inflation has three stages. For this to happen, knowing that $\alpha \sim 1/N_{tot}$, one requires that $1 \ll \beta \ll N_{total}$ so $\beta \sim 10$ or so. On the other hand, if $\kappa_\psi > 1$ then the phase transition is very sharp and $n_p \lesssim 1$. However, due to smallness of α , to have κ_ψ one requires that $\beta \gg N_{tot}$, i.e. $\beta \sim 10^2 - 10^3$.

B. $\phi_c^2 < 12M_P^2$

In this limit, $\psi_{trig}^2 < \psi_{min}^2$ so the back-reaction $g^2\psi^2\phi^2$ becomes important sooner than the self-interaction $\lambda\psi^4$, triggering the inflaton field to roll down quickly to its minimum $\phi = 0$. Thus inflation cannot continue for an extended period in the form of chaotic inflation, and one expects that the completion of symmetry breaking and the end of inflation occurs more or less at the same time. Specifically

$$N_{tot} = N_c + n_{trig} + \Delta n_{\phi-trans} + \Delta n_{\psi-trans} . \quad (2.12)$$

Here n_{trig} denotes the time when the back-reaction of ψ on the inflaton mass become significant, triggering its fast rolling to $\phi = 0$, whereas $\Delta n_{\phi-trans}$ denotes the number of e -folds it takes for the inflaton field to move toward its minimum until the time when the coupling term $g^2\phi^2\psi^2$ becomes negligible compared to the waterfall bare mass term $-M^2\psi^2$, i.e. until when $\phi^2 \ll \phi_c^2$. Since ϕ evolves like $e^{-3n/2}$ this period is rather short, $\Delta n_{\phi-trans} \sim 1$. Once the value of ϕ becomes negligible, the ψ field becomes even more tachyonic and rolls quickly down to its minimum like $\psi \simeq \psi_{trig} e^{\sqrt{\beta}(n-n_{trig})}$.

For large β this period of the ψ transition (symmetry breaking completion) is very quick, $\Delta n_{\psi-trans} \ll 1$. One can show that $n_{trig} \sim \Delta n_{\psi-trans} \sim \frac{1}{\sqrt{\beta}} < 1$, so inflation ends abruptly after the waterfall phase transition as in standard hybrid inflation.

Having specified two major scenarios of hybrid inflation based on the ratio ϕ_c/M_P , we now proceed to the study of curvature perturbations in these scenarios. The case with $\phi_c^2 < 12M_P^2$ is somewhat similar to standard hybrid inflation as studied in [7–11]. We shall come back to this limit in section VI. The case with $\phi_c^2 > 12M_P^2$ is the one that needs careful considerations. In the next three sections, as a sample example, we consider in details a model with $\phi_c^2 > 12M_P^2$ and $\alpha\beta \ll 1$ so $n_p \gg 1$ and inflation has three extended stages. This case perhaps is the most elaborate model in our case study which would be helpful in the classification of hybrid inflation models in section VI. Once we have performed the curvature perturbation analysis in this case, we are able to make a connection to the case $\phi_c^2 < 12M_P^2$ and provide a global view of the issues of curvature perturbations and the waterfall dynamics in different limits of the parameter space of hybrid inflation.

III. BACKGROUND DYNAMICS

The analysis performed in this section and sections IV and V are for the case $\phi_c^2 > 12M_P^2$ and $\alpha\beta \ll 1$. In this section we study the background dynamics in details which is also applicable to other cases.

We assume the standard metric,

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2 , \quad (3.1)$$

where $a(t)$ is the scale factor. As mentioned before, we assume the vacuum energy dominance for most of the inflationary era. Hence we have

$$a(t) \propto e^{Ht} , \quad (3.2)$$

where H is given by (2.3).

In this approximation, the dynamics of the system is solely described by the field equations for ϕ and ψ . The field equations for the classical, homogeneous background fields are written as

$$\phi'' + 3\phi' + \left(\alpha + g^2 \frac{\psi^2}{H^2} \right) \phi = 0, \quad (3.3)$$

$$\psi'' + 3\psi' + \left(-\beta + g^2 \frac{\phi^2}{H^2} + \lambda \frac{\psi^2}{H^2} \right) \psi = 0, \quad (3.4)$$

where the prime denotes the differentiation with respect to n , $' = d/dn$, where $n = N - N_c$, and the dimensionless parameters α and β are introduced in Eq. (2.4). We assume $\alpha \ll 1$ and $\beta \gg 1$. Here and in the following, we denote the classical fields by ϕ and ψ , while the quantum fluctuations by $\delta\phi$ and $\delta\psi$.

At early times $(-n) \gg 1$, the classical waterfall field is trapped at $\psi = 0$ and the quantum fluctuations are completely negligible. Hence if we simply solve the classical field equation (3.4) with the initial condition $\psi = \psi' = 0$, ψ would stay at the origin forever even at $n > 0$. However, because of the exponential growth of the quantum fluctuations of the waterfall field after the transition, the expectation value $\langle \delta\psi^2 \rangle$ becomes non-negligible and the rms value soon starts to behave as a classical field $\psi = \sqrt{\langle \delta\psi^2 \rangle}$. In fact we will show in the next subsection that $\langle \delta\psi^2 \rangle$ is dominated by the superhorizon modes, hence the rms of it is indeed observed as a classical background for an observer within each Hubble horizon region. In particular, in Eqs. (3.3) and (3.4), ψ^2 in the interaction terms $g^2\psi^2/H^2$ and $\lambda\psi^2/H^2$ are to be considered as $\langle \delta\psi^2 \rangle$.

Here we provide the solutions for the ϕ and ψ evolution, starting with the dynamics of ϕ . Let us first assume that the interaction term $g^2\psi^2\phi^2$ is negligible. Namely, we consider the stage when $\alpha \gg g^2\psi^2/H^2$ in Eq. (3.3). With this assumption one can easily solve it to obtain

$$\phi(n) = \phi_c \exp(-r n) \quad (n < n_p), \quad (3.5)$$

with

$$r \equiv \left(\frac{3}{2} - \sqrt{\frac{9}{4} - \alpha} \right) \simeq \frac{\alpha}{3}. \quad (3.6)$$

This stage ends when the value of $g^2\psi^2/H^2$ becomes comparable to α . To simplify the dynamics, we assume that this happens at about the end of the waterfall stage, $n = n_p$, when $\psi^2 = \langle \delta\psi^2 \rangle$ levels off to the local minimum.

Now we look into the dynamics of ψ . Before the transition $\psi = 0$ but it effectively becomes non-vanishing after the transition. During the waterfall stage, we can neglect the self-interaction $\lambda\psi^4$, and the equation for ψ simplifies to

$$\psi'' + 3\psi' + \beta (e^{-2r n} - 1) \psi = 0. \quad (3.7)$$

The general solution is given by a linear combination of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, where $\nu = \frac{\sqrt{\beta+9/4}}{r}$ and $z = \frac{\sqrt{\beta}e^{-rn}}{r}$. However, one can get a better view of the ψ solution using the WKB approximation as we demonstrate below.

Defining $\psi = \Psi e^{-3n/2}$, Eq. (3.7) is rewritten as

$$\Psi'' + \left(-\frac{9}{4} - 2\beta r n + 2\beta r^2 n^2 \right) \Psi = 0, \quad (3.8)$$

in which we expanded the exponential in Eq. (3.7) to second order in $rn \ll 1$. Using the WKB approximation one has

$$\Psi \sim \exp \left[\int dn \sqrt{\frac{9}{4} + 2\beta r n - 2\beta r^2 n^2} \right]. \quad (3.9)$$

As can be seen from Eq. (3.7) (see also Eq. (4.4) below) the sharpness of the phase transition is controlled by the product βr . The larger is βr the greater is the effective tachyonic mass of the waterfall during the phase transition, causing a sharp phase transition. This means that, by keeping the product βr small one basically reduces the effective tachyonic mass of the waterfall field during the phase transition so inflation can continue for an extended period during the course of the phase transition. In the analysis below, we are interested in a situation where $\beta r \ll 1$ so the phase transition is not sharp and $n_p \gg 1$. This is a new limit where the curvature perturbation shows some interesting behavior. However, for this to happen one requires $\beta \ll 1/r$ so β can not be arbitrarily large. For reasonable values

of r this requires $\beta \sim 10$. On the other hand, if one is interested in the limit where $\beta r > 1$, corresponding to $n_p \sim 1$, then the analysis of the curvature perturbation from the waterfall is basically the same as in standard hybrid inflation. We will comment on this case later. In our analysis it is useful to define

$$\kappa_\psi \equiv \frac{\beta r}{3} \simeq \frac{\alpha\beta}{9} \ll 1. \quad (3.10)$$

Calculating the integral in Eq. (3.9) to leading orders in κ_ψ and setting the initial condition at the moment of the phase transition $n = 0$ yields

$$\psi(n) = \psi(0) \exp \left[\kappa_\psi n^2 - \frac{4}{9} \kappa_\psi^2 n^3 \right], \quad (3.11)$$

where $\psi(0) = \sqrt{\langle \delta\psi^2(0) \rangle}$. The good agreement between this solution and the full numerical solution can be seen in Fig. 1. Note that the term in Eq. (3.11) containing $\kappa_\psi^2 n^3$ is much smaller than the term $\kappa_\psi n^2$ so we shall safely ignore it in most of our analysis below.

The above solution is valid until the self interaction term $\lambda\psi^4$ becomes important. Eventually the self-interaction becomes important and ψ settles down to its local minimum because of the large induced mass, $\partial_\psi^2 V(\psi_{min}) = 2(M^2 - g^2\phi^2) \simeq 2M^2$. Here note that the value of ϕ will be a fraction of ϕ_c by the time ψ^2 reaches $\psi_{min}^2(\phi)$ (see the next paragraph) and hence we may assume $M^2 \gg g^2\phi^2$. Denoting the time at which ψ reaches the local minimum by $n = n_p$ we have

$$\psi^2(0) \exp \left[2\kappa_\psi n_p^2 - \frac{8}{9} \kappa_\psi^3 n_p^3 \right] = \psi_{min}^2(\phi) = \frac{M^2}{\lambda} - \frac{g^2\phi^2}{\lambda} \simeq \frac{M^2}{\lambda}, \quad (3.12)$$

or to leading order

$$n_p^2 = -\frac{1}{2\kappa_\psi} \ln \left[\frac{\lambda\psi^2(0)}{M^2} \right]. \quad (3.13)$$

This equation tells us how the total number of e -folds of the waterfall stage depends on $\psi^2(0)$. As discussed in detail in the next section, it will play the central role in the evaluation of the curvature perturbation.

Now, since we have assumed that $g^2\psi^2/H^2$ becomes comparable to α at the end of the waterfall stage, we should examine its effect on the evolution of ϕ . Inserting the solution (3.11) for ψ into Eq. (3.3), one finds instead of Eq. (3.5),

$$\phi(n) = \phi_c \exp \left[-rn - \frac{g^2\psi^2(0)}{3H^2} \int_0^n e^{2\kappa_\psi n'^2} dn' \right]. \quad (3.14)$$

At the end of the waterfall stage, $n = n_p$, we have $\psi^2(n_p) = \psi_{min}^2(\phi) \simeq M^2/\lambda$. Hence the above gives

$$\phi(n_p) = \phi_c \exp \left[-rn_p - \frac{g^2\beta}{3\lambda} \right] = \phi_c \exp \left[-\frac{\alpha}{3}n_p - \frac{\alpha}{3}C \right], \quad (3.15)$$

where we have used Eq. (2.6) in the second equality.

After ψ^2 settles down to $\psi_{min}^2(\phi)$, i.e. at the stage $n > n_p$, we can plug this minimum value into the potential to obtain the effective potential for ϕ ,

$$\begin{aligned} V_{eff}(\phi) &= V(\phi, \psi_{min}(\phi)) = \frac{1}{2}m^2\phi^2 + \frac{g^2}{2\lambda}M^2\phi^2 - \frac{g^4}{4\lambda}\phi^4 \\ &\simeq \frac{1}{2} \left[m^2 + \frac{g^2}{\lambda}M^2 \right] \phi^2, \end{aligned} \quad (3.16)$$

where the second line follows from our assumption that we have $M^2 > g^2\phi^2$ by the end of the waterfall stage. Hence the $g^2\phi^2/2$ term may be ignored at the stage $n > n_p$. The effective mass of ϕ at the stage $n > n_p$ is given by

$$m_{\phi,eff}^2 = m^2 + \frac{g^2}{\lambda}M^2 = m^2(1 + C). \quad (3.17)$$

We note that the potential now behaves exactly the same as in standard chaotic inflation with $m^2 \rightarrow m^2(1 + C)$. In other words, the universe is no longer vacuum-dominated at $n > n_p$ and inflation proceeds as in chaotic inflation.

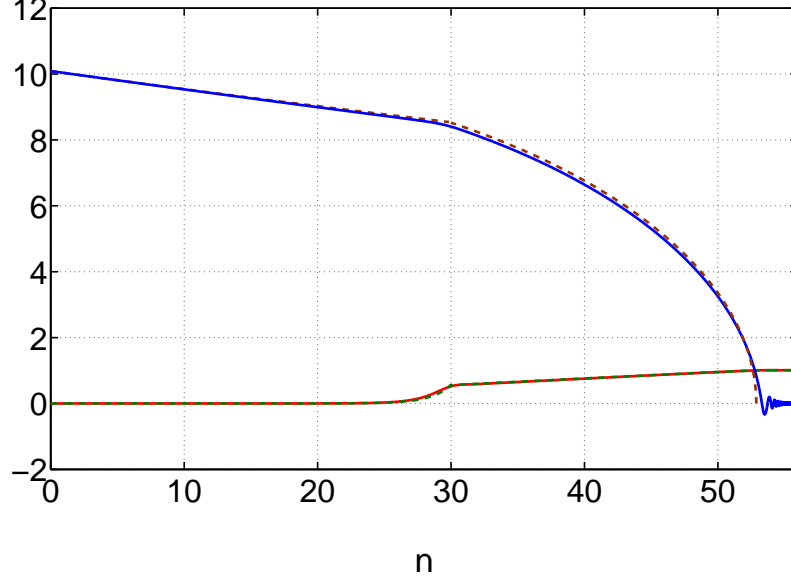


FIG. 1: Here we present the background fields dynamics in units of M_P from the phase transition $n = 0$ to the end of inflation. The solid blue line and the solid red line, respectively, show the full numerical solutions of the ϕ and ψ evolutions with the initial value $\psi(n = 0) = \langle \delta\psi^2(0) \rangle$. The dashed green line and the dashed brown line, respectively, show our approximate analytical solutions for the corresponding fields. The agreement between them is very good. In this picture, the time of phase transition completion is $n_p \simeq 30$. The numeric analysis are for the parameters, $M = 5 \times 10^{-5} M_P$, $m = 2 \times 10^{-6} M_P$, $g^2 = 2 \times 10^{-11}$ and $\lambda = 2 \times 10^{-9}$ corresponding to $\beta \simeq 8.5$ and $\kappa_\psi \simeq 0.02$.

By using the above form of the effective potential, and assuming $\phi(n_p) \gg M_P$, the equation for ϕ at $n > n_p$ reduces to

$$-8M_{pl}^2(n - n_p) = (\phi^2 - \phi_p^2) - (1 + C^{-1}) \phi_c^2 \ln \left[\frac{1 - \frac{\phi_p^2}{\phi_c^2} (1 + C^{-1})^{-1}}{1 - \frac{\phi^2}{\phi_c^2} (1 + C^{-1})^{-1}} \right], \quad (3.18)$$

which for $\phi, \phi_p < \phi_c$ and $C > 1$ can be approximated to

$$\phi^2(n) = \phi_p^2 - 4M_{pl}^2(n - n_p) \quad (n > n_p), \quad (3.19)$$

where $\phi_p \equiv \phi(n_p) = \phi_c e^{-r n_p}$. The slow-roll inflation ends when ϕ becomes of order M_P as usual (approximately at $\phi = \phi_f \simeq \sqrt{2/3} M_P$). Here it should be stressed that the end of inflation is solely determined by the value of ϕ alone. This is an important point to be kept in mind when calculating the curvature perturbation using the δN -formalism [25] (see discussions in Sec. IV C). The overall evolution of the classical fields is depicted in Fig. 1.

It is worth noting again that the above evolutionary picture holds at any horizon patch of the universe. Until the time of transition, $\psi = \sqrt{\langle \delta\psi^2 \rangle}$ cannot be regarded as classical. But as soon as the tachyonic instability sets in, it starts to behave like a classical field. Another important point is that the initial amplitude of these fluctuations at $n = 0$ varies from one patch to another because of the very nature of the quantum fluctuations. As will be discussed in Sec. IV below, this implies the spatial variation of the number of e -folds from the time of transition to the end of the waterfall stage $n = n_p$, which is fixed by the condition $\langle \delta\psi^2(n_p) \rangle = \psi_{min}^2(\phi)$, giving rise to the curvature perturbation at the end of inflation. It may be also noted that the quantum fluctuations of the waterfall field play the role of entropy perturbations if we regard ϕ as the inflaton field. However, as we have described in the above, since they give rise to a non-vanishing classical background field ψ whose evolution is crucial to the dynamics of inflation, the system is better regarded as a two-field inflation model after the waterfall transition.

IV. CURVATURE PERTURBATION FROM THE WATER-FALL FIELD

In this section we study the quantum fluctuations of the waterfall field $\delta\psi$ and calculate the curvature perturbation from them by using the δN -formalism.

A. Quantum fluctuations

The equation governing fluctuations of the waterfall field in real space is

$$\delta\psi'' + 3\delta\psi' - \frac{1}{a^2 H^2} \nabla^2 \delta\psi + \left(-\beta + g^2 \frac{\phi^2}{H^2} + 3\lambda \frac{\psi^2}{H^2} \right) \delta\psi = 0, \quad (4.1)$$

where ψ is the classical background. As we discussed in the previous section, we eventually identify it as $\psi^2 = \langle \delta\psi^2 \rangle$. At early times before the transition or even after the transition, the self interaction term $\lambda \langle \delta\psi^2 \rangle / H^2$ in the mass term of Eq. (4.2) is negligible until the end of the waterfall stage. In this approximation Eq. (4.1) in momentum space \mathbf{k} is given by

$$\delta\psi''_{\mathbf{k}} + 3\delta\psi'_{\mathbf{k}} + \left(\frac{k^2}{a^2 H^2} - \beta + g^2 \frac{\phi^2}{H^2} \right) \delta\psi_{\mathbf{k}} = 0, \quad (4.2)$$

where

$$\delta\psi_{\mathbf{k}} = \int \frac{d^3x}{(2\pi)^{3/2}} \delta\psi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} = \left(a_{\mathbf{k}} \psi_{\mathbf{k}}(n) + a_{-\mathbf{k}}^\dagger \overline{\psi_{\mathbf{k}}(n)} \right). \quad (4.3)$$

Here $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are the annihilation and creation operators, respectively, with respect to a suitably chosen vacuum and $\psi_{\mathbf{k}}(n)$ is the positive frequency function.

Substituting the solution (3.5) for ϕ , one obtains

$$\delta\psi''_{\mathbf{k}} + 3\delta\psi'_{\mathbf{k}} + \left(\frac{k_c^2}{k_c^2} e^{-2n} + \beta (e^{-2rn} - 1) \right) \delta\psi_{\mathbf{k}} = 0, \quad (4.4)$$

where k_c is the comoving wavenumber that crosses the horizon at the critical point $n = 0$, $k_c = Ha(n = 0)$.

We assume a sufficiently small r such that $r|n| \ll 1$ for all scales k of cosmological interest. Hence we have $\beta|e^{-2rn} - 1| \simeq 2\beta r|n|$, which is always smaller than unity because of our assumption that $\kappa_\psi \ll 1$. Hence when the wavenumber k is inside the horizon, $k > k_c e^n$, one can neglect the term proportional to β . The general solution in this limit is given in terms of the Hankel functions $H_{3/2}^{(1)}(x)$ and $H_{3/2}^{(2)}(x)$ by

$$\delta\psi_{\mathbf{k}} = e^{-3n/2} \left(c_1 H_{3/2}^{(1)}(x) + c_2 H_{3/2}^{(2)}(x) \right); \quad x \equiv \frac{k}{k_c} e^{-n}. \quad (4.5)$$

As usual, the natural choice of the vacuum is that it approaches the Minkowski mode function in the short distance limit,

$$\delta\psi_{\mathbf{k}} \rightarrow \frac{e^{-ik\tau}}{a\sqrt{2k}} \quad \text{as } k\tau \rightarrow -\infty, \quad (4.6)$$

where τ is the conformal time, $d\tau = dt/a$. With this initial condition the positive frequency function for the modes inside the horizon is obtained to be

$$\delta\psi_{\mathbf{k}}(n) = \sqrt{\frac{\pi H^2}{4k_c^3}} e^{-3n/2} H_{3/2}^{(1)} \left(\frac{k}{k_c} e^{-n} \right). \quad (4.7)$$

At the time of horizon crossing $n = n_k$ when $e^{n_k} = k/k_c$, the above expression gives

$$|\delta\psi_{\mathbf{k}}(n_k)| \simeq \frac{H}{\sqrt{2k^3}}. \quad (4.8)$$

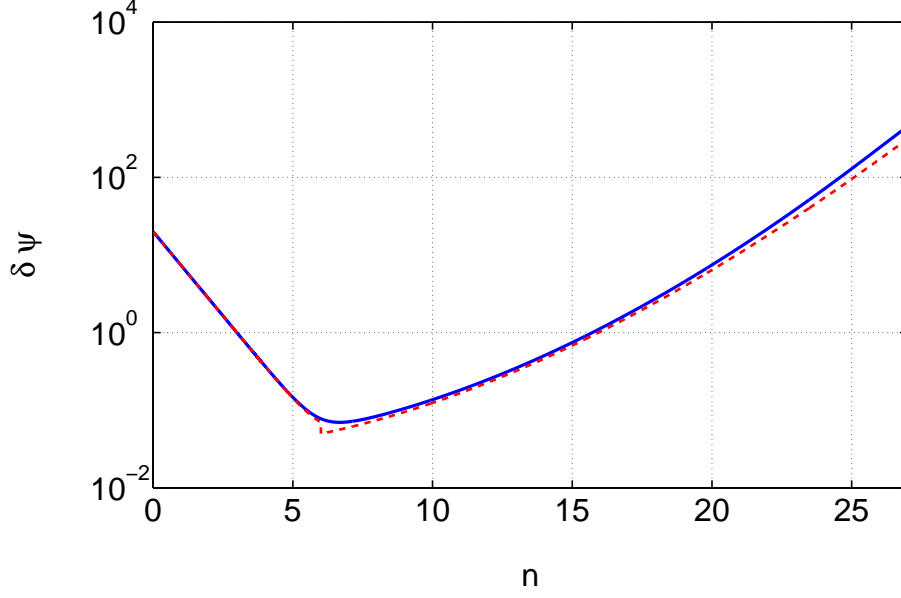


FIG. 2: The amplitude of the mode function $|\psi_k(n)|$ as a function of the number of e-folds n is shown for a mode which leaves the horizon at $n = 6$. The solid blue line is obtained from the full numerical analysis whereas the dashed red line is from our analytical expression (4.9). All numerical parameters are the same as in Fig. 1.

After horizon crossing one can neglect the spatial gradient term $(k/k_c)^2 e^{-2n}$ in Eq. (4.4) and the equation for ψ_k becomes identical to that for the classical background, Eq. (3.7). Hence the solution takes the form (3.11). Matching it with (4.8) at $n = n_k$, we obtain an approximate expression for ψ_k at $n > n_k$ as

$$\delta\psi_k(n) = \frac{H}{\sqrt{2k^3}} \exp \left[\kappa_\psi (n^2 - n_k^2) - \frac{4}{9} \kappa_\psi^2 (n^3 - n_k^3) \right]. \quad (4.9)$$

One can see a good agreement of this analytical expression with the numerical result in Fig. 2.

B. Waterfall stage

Given the solution to the mode functions, an important quantity to compute is the squared fluctuation $\delta\psi^2$, because it determines the duration of the waterfall stage. It is given by

$$\begin{aligned} \delta\psi^2(n, \mathbf{x}) &= \delta\psi^2(0, \mathbf{x}) \exp \left[2\kappa_\psi n^2 - \frac{8}{9} \kappa_\psi^2 n^3 \right]; \\ \delta\psi^2(0, \mathbf{x}) &= \iint_{k, k' < aH} \frac{d^3k d^3k'}{(2\pi)^3} \delta\psi_{\mathbf{k}}(0) \delta\psi_{\mathbf{k}'}^\dagger(0) e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{x}}, \end{aligned} \quad (4.10)$$

where

$$\delta\psi_{\mathbf{k}}(0) = \psi_{\mathbf{k}}(0) A_{\mathbf{k}} = \frac{H}{\sqrt{2k^3}} e^{-\kappa_\psi n_k^2 + \frac{4}{9} \kappa_\psi n_k^3} A_{\mathbf{k}}; \quad A_{\mathbf{k}} \equiv \left(a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger \right), \quad (4.11)$$

and we have absorbed an irrelevant phase factor in the definitions of $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}^\dagger$. Note that the integral is cut off at the horizon scale because the modes on subhorizon scales are just the standard vacuum fluctuations which are to be regularized to zero.

The above expression for $\delta\psi_{\mathbf{k}}$ shows that it is a classical random fluctuation, because it is proportional to $A_{\mathbf{k}}$ which commutes with $A_{\mathbf{p}}^\dagger (= A_{-\mathbf{p}})$ for any pair of \mathbf{k} and \mathbf{p} . Hence for each horizon-size patch one would observe $\delta\psi^2$ as a homogeneous classical background which varies smoothly over scales larger than the horizon scale. Thus on a given,

sufficiently large scale, say the comoving scale of the present Hubble horizon size, one can calculate the mean value $\langle \delta\psi^2(n) \rangle$ and the fluctuation

$$\Delta\psi^2(n, \mathbf{x}) \equiv \delta\psi^2(n, \mathbf{x}) - \langle \delta\psi^2(n) \rangle, \quad (4.12)$$

where $\langle \delta\psi^2(n) \rangle$ determines the homogeneous background while $\Delta\psi^2(n, \mathbf{x})$ gives rise to the curvature perturbations on superhorizon scales.

Let us first consider the background dynamics. Taking the expectation value of $\delta\psi^2$, we find

$$\langle \delta\psi^2(n) \rangle = \langle \delta\psi^2(0) \rangle \exp[2\kappa_\psi n^2 - \frac{8}{9}\kappa_\psi^2 n^3], \quad (4.13)$$

where

$$\langle \delta\psi^2(0) \rangle = \int_{k < aH} \frac{d^3k}{(2\pi)^3} |\psi_k(0)|^2 = \left(\frac{H}{2\pi} \right)^2 \int_{k < aH} \frac{dk}{k} \exp[-2\kappa_\psi n_k^2 + \frac{8}{9}\kappa_\psi^2 n_k^3]. \quad (4.14)$$

By changing the variable of integration to $n_k = \ln(k/k_c)$, we may approximate the upper and lower limits of the integral to plus and minus infinity, respectively, for $\kappa_\psi n^2 \gg 1$, and neglect the term $\kappa_\psi^2 n_k^3$ in the exponent, to obtain

$$\langle \delta\psi^2(0) \rangle \simeq \left(\frac{H}{2\pi} \right)^2 \int_{-\infty}^{\infty} dn_k \exp(-2\kappa_\psi n_k^2) = \frac{H^2}{4\pi^2} \sqrt{\frac{\pi}{2\kappa_\psi}}. \quad (4.15)$$

The waterfall stage ends when the self-interaction term becomes important and $\langle \delta\psi^2 \rangle$ reaches the local minimum $\psi_{min}^2(\phi)$ given by Eq. (3.12). Thus we have

$$\langle \delta\psi^2(n_p) \rangle = \langle \delta\psi^2(0) \rangle \exp[2\kappa_\psi n_p^2 - \frac{8}{9}\kappa_\psi^2 n_p^3] = \psi_{min}^2(\phi). \quad (4.16)$$

Neglecting the $\kappa_\psi^2 n_p^3$ term, we may approximately solve this for n_p to find

$$n_p^2 \simeq -\frac{1}{2\kappa_\psi} \ln \left(\frac{\langle \delta\psi^2(0) \rangle}{\psi_{min}^2(\phi)} \right) \simeq -\frac{1}{2\kappa_\psi} \ln \left(\frac{\beta}{\lambda} \sqrt{32\pi^3 \kappa_\psi} \right). \quad (4.17)$$

Before closing this subsection, let us make an important comment concerning the meaning of Eq. (4.11). If we would literally consider it as $\delta\psi^2(n, \mathbf{x})$ at $n = 0$, then it could only take account of the comoving wavenumbers k smaller than k_c . However, as can be seen from Eq. (4.10), it takes account of all the comoving wavenumbers which has crossed the horizon at $n = n_k$. This implies that $\delta\psi^2(0, \mathbf{x})$ introduced in Eq. (4.11) is defined just for convenience to express $\delta\psi^2(n, \mathbf{x})$ at the end of the waterfalls stage $n = n_p$.

C. δN -formalism

We now evaluate the curvature perturbation from inflation by using the δN -formalism [25]. In this formalism, one first calculates the number of e -folds for the background homogeneous universe from a fixed final epoch $t = t_f$ backward in time to an arbitrary initial epoch t , $N(t \rightarrow t_f)$. The final epoch t_f must be chosen such that at and after $t = t_f$ the evolutionary trajectory of the universe is unique.

Then one considers the perturbation δN by identifying the final time slice to be a comoving (or uniform density) hypersurface and the initial slice to be a flat hypersurface. Naturally there will be fluctuations in the matter fields on the initial flat slice which produce δN . Then the resulting δN is equal to the comoving curvature perturbation on the final hypersurface $\mathcal{R}_c(\mathbf{x}, t_f) = \delta N(\mathbf{x})$. In the present model, we assume that the evolution of the universe is unique, i.e. there is no isocurvature perturbation, at and after the end of inflation. As before we use n as the time instead of t in what follows. Hence in particular, $N = n - n_f$ where n_f is the number of e -folds from the critical point until the end of inflation.

There are a couple of points to be kept in mind when evaluating δN in the present model. The first point is about the dependence of N on the fields ϕ and ψ . Since we have to relate the fluctuations in these fields to δN , we have to express N in terms of ϕ and ψ . Namely, we need an expression like $N = N(\phi(n), \psi(n))$.

Going backward in time from the end of inflation up to the end of the waterfall stage $n_p < n < n_f$, since the value of the waterfall field is determined by the value of the inflation, $\psi^2 = \psi_{min}^2(\phi)$, with the identification $\psi = \sqrt{\langle \delta\psi^2 \rangle}$,

N is a function of only ϕ , $N = N(\phi)$. Since the scales that leave the horizon after the waterfall stage $n > n_p$ are presumably too small to be of cosmological interest, we focus on the stage $n < n_p$ in the following discussion.

Here the very important point is that the time when the waterfall stage ends, $n = n_p$, depends heavily on the initial value of ψ at the beginning of the waterfall stage, $\psi^2(n = 0)$, hence on the amplitude of the fluctuations $\langle \delta\psi^2(0) \rangle$. This implies that N is a function of both ϕ and ψ^2 through its dependence on n_p ,

$$N = N(\phi(n), \psi^2(n)) \quad \text{at } n < n_p. \quad (4.18)$$

The second point is about the choice of the initial hypersurface. In the present model the comoving curvature perturbation due to $\delta\phi$ is conserved on superhorizon scales. Therefore, instead of choosing the time of horizon crossing for each comoving scale, we can take the initial slice to be that at the time of transition $n = 0$ but with $\delta\phi$ given by that on the flat hypersurface at its horizon crossing as given by Eq. (4.21) below. Therefore for ϕ -dependence in N , we have

$$\delta_\phi N \equiv N(\phi_c + \delta\phi_c, \psi) - N(\phi_c, \psi) = N_\phi(\phi + \delta\phi \rightarrow \phi_c) - N_\phi(\phi \rightarrow \phi_c), \quad (4.19)$$

where $N_\phi(\phi \rightarrow \phi_c)$ is the number of e -folds from the time when the inflaton had a value ϕ until the time of transition when $\phi = \phi_c$, and $\delta\phi$ and $\delta\phi_c$ are, respectively, the values of the field fluctuation at horizon crossing and at the time of transition. The contribution $\delta_\phi N$ may be approximated to linear order in $\delta\phi$ as it is the same as the conventional slow-roll single-field contribution which contains negligible non-Gaussianity. Noting

$$\frac{\partial N}{\partial \phi} \delta\phi = -\frac{H}{\dot{\phi}} \delta\phi = -\frac{\delta\phi}{\phi'} = \frac{1}{r} \frac{\delta\phi}{\phi}, \quad (4.20)$$

where we have used Eq. (3.5), we can evaluate $\delta_\phi N$ as

$$\delta_\phi N = \frac{1}{r} \frac{\delta\phi_c(\mathbf{x})}{\phi_c} = \frac{1}{r} \frac{\delta\phi(n, \mathbf{x})}{\phi(n)} = \frac{e^{rn}}{r} \frac{\delta\phi(n, \mathbf{x})}{\phi_c}, \quad (4.21)$$

where we have used the fact that $\delta\phi$ on superhorizon scales has the same time dependence as the background ϕ . When evaluating the spectrum, as usual, for a given scale k we choose n to be the time of horizon crossing $n = \ln(k/k_c)$ at which we have $\langle \delta\phi_k^2 \rangle = H^2/(2\pi)^2$.

The contribution from $\delta\psi$ can be taken into account by taking the difference between the number of e -folds with $\delta\psi^2$ and $\psi^2 = \langle \delta\psi^2 \rangle$,

$$\delta_\psi N = N(\phi_c, \delta\psi^2) - N(\phi_c, \langle \psi^2 \rangle), \quad (4.22)$$

where $\delta\psi^2$ and $\langle \delta\psi^2 \rangle$ are to be evaluated at the time of transition $n = 0$. Rigorously speaking, for scales with $k > k_c$, it should be evaluated at its horizon crossing $n = n_k(> 0)$. However, as noted at the end of the previous subsection, we have conveniently included the modes $k > k_c$ also on the hypersurface at $n = 0$ in $\delta\psi^2(0, \mathbf{x})$ by formally extending their evolutionary behavior back to $n = 0$ as if they were already outside the horizon. Hence one may calculate $\delta_\psi N(\mathbf{x})$ by just considering the fluctuations in $\delta\psi^2(0, \mathbf{x})$ without separating the modes into the two ranges $k < k_c$ and $k > k_c$. A justification of this point is discussed in more detail in Appendix A.

Equation (4.17) shows how the number of e -folds from the time of transition until the end of the waterfall stage depends on the magnitude of the initial fluctuations $\langle \delta\psi^2(0) \rangle$. This implies that the total number of e -folds of the waterfall stage fluctuates from one horizon patch to another due to the fluctuations in $\delta\psi^2(0, \mathbf{x})$, giving rise to the curvature perturbation. Namely, we have

$$\left(1 + \frac{\delta n_p(\mathbf{x})}{n_p}\right)^2 = 1 - \frac{1}{2\kappa_\psi n_p^2} \ln \left(\frac{\delta\psi^2(0, \mathbf{x})}{\langle \delta\psi^2(0) \rangle} \right), \quad (4.23)$$

where $\delta n_p(\mathbf{x})$ is the spatial fluctuation in the number of e -folds from the time of transition until the end of the waterfall stage. Apparently we have $\delta_\psi N = \delta n_p(\mathbf{x})$.

In the limit $\delta n_p/n_p \ll 1$, we can linearize the above to obtain

$$\delta_\psi N = \delta n_p = \frac{-1}{4\kappa_\psi n_p} \ln \left(\frac{\delta\psi^2(0, \mathbf{x})}{\langle \delta\psi^2(0) \rangle} \right). \quad (4.24)$$

One may further expand the above logarithm, by using $\ln(1+x) = x - x^2/2 + \dots$, to find

$$\delta_\psi N = \frac{-1}{4\kappa_\psi n_p} \left[\frac{\Delta\psi^2(0, \mathbf{x})}{\langle \delta\psi^2(0) \rangle} - \frac{1}{2} \left(\frac{\Delta\psi^2(0, \mathbf{x})}{\langle \delta\psi^2(0) \rangle} \right)^2 + \dots \right], \quad (4.25)$$

where, following Eq. (4.12), we have $\Delta\psi^2(0, \mathbf{x}) = \delta\psi^2(0, \mathbf{x}) - \langle\delta\psi^2(0)\rangle$.

Here a comment is in order. One might worry that there could be a contribution to δn_p from the fluctuations in ϕ through the ϕ -dependence of ψ_{min}^2 . Indeed there exists some contribution because the value of ϕ at the end of the waterfall stage depends on the number of e -folds from the critical point as $\phi(n_p) \simeq \phi_c e^{-r n_p}$. However, one can easily show that this contribution adds a term $-r\delta n_p$ to the terms inside the square brackets in Eq. (4.25), hence gives a negligible correction for $\kappa_\psi n_p/r \sim \beta n_p \gg 1$.

After the completion of the waterfall stage the trajectory in the two-dimensional field space (ϕ, ψ) is essentially unique, with ψ being completely subject to ϕ , as can be seen from Fig. 3, and the end of inflation is synchronized on the uniform ϕ hypersurface $\phi = \phi_f$ as in the case of single-field slow-roll inflation. This implies that there will be no contribution to the total δN from the era after the waterfall stage.

As we mentioned before, the field trajectory is unique after the waterfall stage until the end of inflation. Setting $n = n_f$ and $\phi^2(n_f) = \phi_f^2$ in Eq. (3.19), the number of e -folds for this period $\Delta n_{ch} \equiv n_f - n_p$ can be obtained as

$$n_f - n_p = \frac{\phi_p^2 - \phi_f^2}{4M_{pl}^2}. \quad (4.26)$$

Hence the fluctuation in the number of e -folds from this slow-roll stage, $\delta(\Delta n_{ch})$ is given by

$$\delta(\Delta n_{ch}) = \frac{\delta(\phi_p^2)}{4M_{pl}^2} \simeq -r \frac{\phi_c^2}{2M_{pl}^2} \delta n_p. \quad (4.27)$$

As is clear from this expression, $\delta(\Delta n_{ch})$ is suppressed by the factor r ($\ll 1$) relative to δn_p .

To summarize, the contribution to δN after the transition is equal to $\delta_\psi N$ to a good approximation. Hence the total δN is expressed as

$$\delta N = \delta_\phi N + \delta_\psi N, \quad (4.28)$$

where $\delta_\phi N$ is given by Eq. (4.21) and $\delta_\psi N$ is given approximately by Eq. (4.25). We note that, if we express Eq. (4.25) in the form

$$\delta_\psi N = N_{,\psi^2} \Delta\psi^2 + \frac{1}{2} N_{,\psi^2, \psi^2} (\Delta\psi^2)^2 + \dots, \quad (4.29)$$

we have the relation among the coefficients,

$$N_{,\psi^2} = -\langle\delta\psi^2\rangle N_{,\psi^2, \psi^2} = \frac{-1}{4\kappa_\psi n_p} \frac{1}{\langle\delta\psi^2\rangle}. \quad (4.30)$$

This is a useful relation when one wants to evaluate the non-Gaussianity in this model.

V. POWER SPECTRUM AND CORRELATION FUNCTIONS

This section is devoted to the calculations of the power spectrum and correlation functions of the curvature perturbation. We find that the curvature perturbation induced from the waterfall is too large to be normalized to the COBE/WMAP normalization.

A. Power Spectrum and spectral tilt

The most important result of the δN formalism is that one can identify the curvature perturbation on the final comoving surface $\mathcal{R}_c(\mathbf{x})$, which is conserved thereafter, by the fluctuation in the number of e -folds from an initial flat surface to the final comoving surface, $\mathcal{R}_c(\mathbf{x}) = \delta N(\mathbf{x})$. Taking the Fourier component of Eq. (4.28) with the help of Eqs. (4.21) and (4.25), to leading order we obtain

$$\mathcal{R}_\mathbf{k} = -\frac{1}{4\kappa_\psi n_p} \frac{\Delta\psi_\mathbf{k}^2(0)}{\langle\delta\psi^2(0)\rangle} + \frac{1}{r} \left(\frac{k}{k_c}\right)^r \frac{\delta\phi_\mathbf{k}(n_k)}{\phi_c}. \quad (5.1)$$

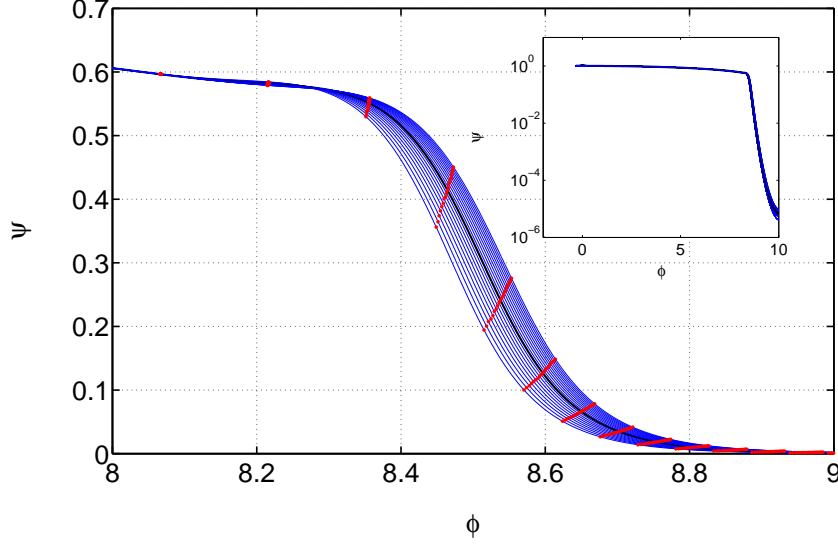


FIG. 3: Larger figure: Contour plot of constant N surfaces in the field-space are shown where ψ and ϕ are in M_P units. Solid blue lines correspond to different patches of the Universe with different initial amplitude of the quantum fluctuation $\delta\psi^2(0, \mathbf{x})$ at the critical point. The solid black curve denotes the trajectory with $\delta\psi^2(0, \mathbf{x}) = \langle \delta\psi^2 \rangle$. This figure shows the time interval between the critical point and few e-folds after the completion of the transition. Red contours show the constant N contours. Counting the number of e -folds from the end of inflation, once the quantum back-reactions of $\delta\psi$ become important the constant N contours incline towards the constant ψ lines while during the earlier stage of inflation the constant N lines coincide with constant ϕ lines. Smaller (inner) figure: Trajectories of different patches of the Universe with different initial amplitudes of quantum fluctuations $\delta\psi^2(0, \mathbf{x})$ at the critical point. This figure provides an extended range for the interval between the critical point and the end of inflation. The numerical parameters are the same as in Fig. 1.

Using the fact that $(\Delta\psi^2)_{\mathbf{k}} = (\delta\psi^2)_{\mathbf{k}}$, the two point correlation function of the curvature perturbations can be read as

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = \frac{1}{16\kappa_\psi^2 n_p^2} \frac{\langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \rangle_0}{\langle \delta\psi^2 \rangle_0^2} + \frac{1}{r^2} \left(\frac{k}{k_c} \right)^{2r} \frac{\langle \delta\phi_{\mathbf{k}} \delta\phi_{\mathbf{k}'} \rangle_{n_k}}{\phi_c^2}, \quad (5.2)$$

where the second term is the contribution of the original inflaton field and the first term is the induced curvature perturbation from the waterfall field.

To calculate the power spectrum of the curvature perturbation we first start off by calculating the correlation function of $\delta\psi^2$ at the critical point, $n = 0$. As shown in [10], this correlation takes the form,

$$\begin{aligned} \langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \rangle_0 &\equiv P_{\delta\psi^2}(k) (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \\ &= 2 \int d^3q |\delta\psi_q(0)|^2 |\delta\psi_{|\mathbf{k}-\mathbf{q}|}(0)|^2 \delta^3(\mathbf{k} + \mathbf{k}'). \end{aligned} \quad (5.3)$$

Here we estimate this integral which would suffice for our discussion here. In Appendix B we provide a more rigorous treatment of calculating this integral. Let us first estimate the contribution of the infrared modes in this integral, say $|q| \lesssim |k|$. The contribution of the long wavelength modes to the two point correlation function approximately can be found to be

$$P_{\delta\psi^2}^{IR} = \frac{2|\delta\psi_k(0)|^2}{(2\pi)^3} \int_{|q| \lesssim |k|} d^3q |\delta\psi_q(0)|^2. \quad (5.4)$$

The above integral counts the effects of large modes in the convolution. Note that we take the term $|\delta\psi_{|\mathbf{k}-\mathbf{q}|}(0)|^2$ out of the integral which is possible for small q . To simplify the analysis, one can take the upper bound to be k_c so the contribution of the infra red momenta in the convolution can be read as

$$P_{\delta\psi^2}^{IR} = |\delta\psi_k(0)|^2 \frac{H^2}{4\pi^2} \int_{-\infty}^0 dn_q e^{-2\kappa_\psi n_q^2} = \frac{1}{2} \langle \delta\psi^2(0) \rangle |\delta\psi_k(0)|^2. \quad (5.5)$$

Now we estimate the contribution of the small modes in the convolution of integral in Eq. (5.3). By the small modes we mean the modes for which, $|\mathbf{q}| > |\mathbf{k}|$. By this assumption, one can simply find

$$P_{\delta\psi^2}^{UV} = \frac{1}{(2\pi)^3} \int_{|\mathbf{q}| \gtrsim |\mathbf{k}|} d^3q |\delta\psi_q(0)|^4 \simeq \frac{H^4}{16\pi^2 k_c^3} \int_{\ln k/k_c}^{\infty} dn_q e^{-3n_q - 4\kappa_\psi n_q^2}. \quad (5.6)$$

As we are interested in the $|\mathbf{q}| \gtrsim |\mathbf{k}|$ to lower bound simply taken to be $k = q$ in r.h.s. The second order term in the exponent suggests a natural cut-off $n_q < 1/\sqrt{2\kappa_\psi}$ for the integral so one can simply ignore the quadratic term of n_q in the exponent to obtain

$$P_{\delta\psi^2}^{UV} \sim \frac{\sqrt{\kappa_\psi}}{2} \langle \delta\psi^2(0) \rangle |\delta\psi_k(0)|^2. \quad (5.7)$$

One readily finds that the ultraviolet contribution from Eq. (5.7) is suppressed compared to the infrared contribution, Eq. (5.5), by a factor $\sqrt{\kappa_\psi} \ll 1$ so the power spectrum from the ψ contribution mainly comes from the infrared modes and $P_{\delta\psi^2} \simeq P_{\delta\psi^2}^{IR}$.

Finally from Eq. (5.2) the curvature perturbation spectrum per unit logarithmic frequency interval,

$$\mathcal{P}_{\mathcal{R}} \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}} = \frac{k^3}{2\pi^2} |\mathcal{R}_k|^2 \quad (5.8)$$

is obtained to be

$$\begin{aligned} \mathcal{P}_{\mathcal{R}} &\simeq \mathcal{P}_{\mathcal{R}}^{IR} + \mathcal{P}_{\mathcal{R}}^{(\phi)} \\ &= \left(\frac{1}{4\kappa_\psi n_p} \right)^2 \frac{k^3}{4\pi^2} \frac{|\delta\psi_k(0)|^2}{\langle \delta\psi^2(0) \rangle} + \frac{g^2}{4\pi^2 r^2 \beta} e^{2rn_k} \\ &\simeq \left(\frac{1}{4\kappa_\psi n_p} \right)^2 \sqrt{\frac{\kappa_\psi}{2\pi}} e^{-2\kappa_\psi n_k^2} + \frac{g^2}{4\pi^2 r^2 \beta} e^{2rn_k}. \end{aligned} \quad (5.9)$$

We need $g \ll 1$ so the contribution from the inflaton field, $\mathcal{P}_{\mathcal{R}}^{(\phi)}$ is sub-leading compared to $\mathcal{P}_{\mathcal{R}}^{IR}$.

With the curvature perturbation power spectrum dominated by $\mathcal{P}_{\mathcal{R}}^{IR}$ the spectral index of the curvature perturbation can be easily found as

$$n_s - 1 = \frac{d \ln(\mathcal{P}_{\mathcal{R}})}{d \ln k} = -4\kappa_\psi n_k = -4\kappa_\psi \ln(k/k_c). \quad (5.10)$$

Since $\kappa_\psi \ll 1$, the curvature perturbation is nearly scale invariant but it has a slight running. It runs from blue for modes which exit the horizon before the transition to red for modes which exit the horizon after the transition. One can easily quantify this running as

$$\frac{dn_s}{d \ln k} = -4\kappa_\psi. \quad (5.11)$$

This should be compared with the standard hybrid inflation where it is found that $n_s \simeq 4$ [7–11], a very blue-tilted spectrum. This is the key property which renders the curvature perturbations from the waterfall field harmless in standard hybrid inflation.

Now we face the fatal problem in our model. We see from Eq. (5.9) that the curvature perturbation power spectrum cannot be normalized to the required COBE/WMAP normalization $\mathcal{P}_{\mathcal{R}} \sim 2 \times 10^{-9}$. With $\kappa_\psi \ll 1$ and $4\kappa_\psi n_p \sim 1$ one finds that $\mathcal{P}_{\mathcal{R}}^{IR}$ is many orders of magnitude bigger than the COBE amplitude. In comparison, in standard hybrid inflation this problem does not show up. As shown in recent analysis [7–11] the curvature perturbations from the waterfall field in standard hybrid inflation with $\phi_c \ll M_P$ and $\kappa_\psi \gg 1$ scales like $(k/k_c)^3 \sim e^{-3N_c}$ which is totally negligible on the cosmological scales.

One may think under what conditions the curvature perturbations induced from the waterfall can be kept subdominant on the cosmological scales. To address this let us first estimate the minimum e-foldings required between the transition time and the time of horizon crossing of the large scale (CMB) modes. We would like to impose the following constraint on the amplitude of these modes

$$\mathcal{P}_{\mathcal{R}}^{IR}(k^{CMB}) < 2 \times 10^{-9}, \quad (5.12)$$

in which k^{CMB} is the momentum of cosmological sizes observed on CMB. By using Eq. (5.9), for typical values of our parameters, this yields

$$|n_{k^{CMB}}| > 23 \times \sqrt{\frac{1}{2\kappa_\psi}}. \quad (5.13)$$

To have $50 - 60$ e -folds after the transition, we must have $n_p + \Delta n_{ch} = 50 - 60$. But if Δn_{ch} dominates it is essentially equivalent to a chaotic inflation model which is not interesting. Hence let us assume $n_p \gtrsim 50$ or so. Then Eq. (5.13) above implies that this is possible only for $n_k^{CMB} < 0$. But then the number of e -folds from the CMB scale till the end of inflation becomes too large. Hence it is impossible to hide the curvature perturbation from the waterfall field for the range of the parameters we consider unless the final chaotic inflation stage dominates.

Now comes the natural question that what is the source of the problem? The key to this questions is the lack of a classical background ψ trajectory so the quantum fluctuations $\delta\psi_k(n)$ determine the effective classical trajectory $\sqrt{\langle\delta\psi^2(n)\rangle}$. The ratio of quantum fluctuation $\delta\psi_k$ to this effective background trajectory represents the amplitude of the quantum fluctuations. This ratio in the coordinate space is

$$\frac{k^3}{(2\pi)^3} \frac{|\delta\psi_k(0)|^2}{\langle\delta\psi^2(0)\rangle} \sim \sqrt{\kappa_\psi}, \quad (5.14)$$

which is not small enough to be normalized to the observations. To see this better compare the situation with the standard chaotic models. In models of chaotic inflation, the fluctuations are given by $\delta\phi/\phi$ in which $\delta\phi \sim H$ while the value of the background ϕ field is another free parameter which can be tuned independently to get the right normalization. In other words, in chaotic models there is a well-defined background classical ϕ field which exists independent of the quantum fluctuations $\delta\phi$. However, in our model there is no classical background ψ field. Instead the effective background waterfall field $\sqrt{\langle\delta\psi^2(n)\rangle}$ is constructed from the quantum fluctuations $\delta\psi_k(n)$ and as a result there is no free parameter to tune the relative ratio $k^3|\delta\psi_k(0)|^2/\langle\delta\psi^2(0)\rangle$ to a very small value.

Having this said, one can naturally ask why this problem does not appear in standard hybrid inflation model. As mentioned above, the main feature in standard hybrid inflation is that the large scale waterfall perturbations are highly blue-tilted, scaling like $(k/k_c)^3$. On the other hand, the main contribution into $\sqrt{\langle\delta\psi^2\rangle}$ comes from small scales, modes which become tachyonic but remained sub-horizon till end of inflation. As a result the cosmological large scales, corresponding to the current Hubble size, are suppressed compared to these small scales by the ratio $(k/k_c)^3 \sim e^{-3N_c/2}$ and therefore are completely negligible in the curvature power spectrum. In contrast, in our model the power spectrum is nearly scale invariant, as can be seen from Eq. (5.10), so the large scale modes are not significantly suppressed compared to the small scales and both of these modes contribute equally into $\sqrt{\langle\delta\psi^2\rangle}$. As a result, one cannot tune the ratio $k^3|\delta\psi_k(0)|^2/\langle\delta\psi^2(0)\rangle$ to a very small value.

In passing we note that in hybrid inflation with a sharp phase transition if one reduces N_c so the phase transition happens early during inflation, i.e. $N_c \sim 5$, then the large scale modes may survive the suppression. We address this case in the next section.

VI. GENERAL PARAMETER SPACE OF HYBRID INFLATION

In last three sections we have focused on the model where $\phi_c^2 > 12M_P^2$ and $\kappa_\psi \ll 1$ so inflation has three extended stages where $N_{tot} = N_c + n_p + \Delta n_{ch}$. We have particularly studied this limit of the parameter space in great details for two reasons. First, this limit was not studied previously in the literature, namely the effects of the waterfall phase transition on curvature perturbations. Secondly, this limit of parameter space provides us a specific example in which one finds an unexpected result that the contribution of the waterfall field to the power spectrum is too big to satisfy the COBE normalization. As we have seen in the previous section, this is because the effective background trajectory $\langle\delta\psi^2\rangle$ gets more or less equal contributions from both the large scale and small scale quantum fluctuations $\delta\psi_k$ and as a result there is no free parameter to suppress the ratio $|\delta\psi_k(0)|^2/\langle\delta\psi^2(0)\rangle$.

For the other limit of parameter space, we have standard hybrid inflation with $\phi_c < M_P$ and $\kappa_\psi > 1$ so inflation ends quickly after the phase transition. As shown in [7–11] the induced curvature perturbation from the waterfall field is highly blue-tilted and is completely negligible on large scales. We would like to search the parameter space of hybrid inflation in general and examine the dynamics of waterfall phase transition and its contribution to power spectrum. We would like to see under what conditions the results vary between the results of standard hybrid inflation and the results we found in the previous sections.

As mentioned in section II, two major classes of the parameter space were determined by the ratio $\phi_c^2/12M_P^2$. In the previous sections we have studied the case $\phi_c^2 > 12M_P^2$ and $\kappa_\psi \ll 1$ in details. Now it is time to consider other limits of the parameter space.

$$\text{A.} \quad \phi_c^2 > 12M_P^2 \quad , \quad \kappa_\psi \gg 1$$

Unlike the case studied in the previous sections, the condition $\kappa_\psi \gg 1$ results in a sharp phase transition similar to standard hybrid inflation. However, in contrast to standard hybrid inflation in which inflation ends shortly after the waterfall, in this case with $\phi_c^2 > 12M_P^2$ inflation continues for an extended period after the transition. On the other hand, knowing that $\alpha \sim 1/N_{tot} \ll 1$, to have $\kappa_\psi \gg 1$ one requires that $\beta \gg N_{tot}$, i.e. $\beta \sim 10^2 - 10^3$. As in previous sections, inflation has three stages and $N_{total} = N_c + n_p + \Delta n_{ch}$. Here we first briefly review the background dynamics and then compute the final curvature perturbations using the δN formalism.

Stage.1 The first stage corresponds to time before phase transition, i.e. $N < N_c$ when the inflaton field reaches to the critical value, $\phi = \phi_c$, after which the waterfall field becomes tachyonic. Fluctuations of the inflaton field, $\delta\phi$, which exit the horizon before the critical point cause variations on the duration of this period, δN_c . By using Eq. (3.5), one has $\phi_c = \phi_i e^{-r n_i}$, in which ϕ_i and n_i respectively are the value of the inflaton field and the time of horizon crossing for a specific mode. So the variation of N_c can be read as

$$\delta N_c = \frac{-1}{r} \left(\frac{\delta\phi}{\phi} \right)_{k < k_c} \quad (6.1)$$

Since the next stage of inflation starts with $\phi = \phi_c$ for each patch therefore the information associated to these perturbations is encoded in δN_c

Stage.2 This stage is defined as the period after the transition until the completion of transition, n_p , when ψ becomes

$$\langle \delta\psi^2 \rangle = \psi_{min}^2(\phi) \equiv \frac{M^2}{\lambda} - \frac{g^2}{\lambda} \phi^2. \quad (6.2)$$

As in standard hybrid inflation this period is very short, $n_p \lesssim 1$, but since the inflaton field is super-Planckian inflation can resume for the third stage in the form of chaotic inflation. As shown in [8–11] the spectrum of the waterfall fluctuations which become tachyonic shortly after the transition is highly blue-tilted while for modes which become tachyonic later on the power spectrum is highly red-tilted. For modes in between the spectral index is running significantly from blue to red so one should avoid these modes in contributing to the total power spectrum. As we discussed before the amplitude of cosmological large scales is suppressed by the factor $e^{-3N_c/2}$ compared to the small scales. Since the small scale modes have the dominant contributions into $\sqrt{\langle \delta\psi^2 \rangle}$ as the effective classical trajectory, the contribution of the large scales is suppress in the power spectrum by the ratio e^{-3N_c} . Below we will provide a bound on N_c in which one can safely neglect the contribution of large scale curvature perturbations.

Stage.3 This stage is the period of chaotic inflation after the transition, $n > n_p$. The end of this period is defined by an explicit value of the ϕ field. The ψ field in this period does not play any role because it is held on its minimum. The inflaton fluctuations which exit the horizon after critical point can change the number of e -folds during chaotic period. From Eq. (4.26) one has

$$n_f - n_p = \frac{\phi_p^2 - \phi_f^2}{4M_{pl}^2}. \quad (6.3)$$

To calculate curvature perturbations using δN formalism first concentrate on the fluctuations which exit the horizon before the critical point. One can borrow the results from Section IV C. The only difference is the dynamics of the waterfall field. In general suppose that the waterfall field has the following form after the transition

$$\psi^2(n) = \psi^2(0) e^{2f(n)}. \quad (6.4)$$

For our case at hand, in which the phase transition happens similar to standard hybrid inflation [8, 10], we have

$$f(n) = \frac{2}{3} \sqrt{6\kappa_\psi} n^{3/2}. \quad (6.5)$$

The time at the end of transition can be found by using the equation,

$$\psi_{min}^2 = \delta\psi^2(0) e^{2f(n_p)}. \quad (6.6)$$

As before, ignoring the slight dependence of $\psi_{min}(\phi)$ on ϕ , the variations of the phase transition duration to the first order can be read as

$$\delta n_p(\mathbf{x}) \simeq \frac{-1}{2 f'(n_p)} \frac{\Delta\psi^2(\mathbf{0}, \mathbf{x})}{\langle \delta\psi^2(0) \rangle}, \quad (6.7)$$

in which $\Delta\psi^2(0, \mathbf{x}) = \delta\psi^2(0, \mathbf{x}) - \langle \delta\psi^2(0) \rangle$.

Adding up the contributions from the inflaton field, Eq. (6.1), and the waterfall field, Eq. (6.7), one finds the final curvature perturbation for modes $k < k_c$,

$$\mathcal{R}_c(k) = \delta N = \frac{-1}{2 f'(n_p)} \frac{(\delta\psi^2)_k}{\langle \delta\psi^2(0) \rangle} + \left(\frac{\delta\phi}{r\phi} \right)_k; \quad k < k_c. \quad (6.8)$$

Finding the curvature perturbation for modes which leave the horizon after waterfall, $k > k_c$, requires some careful considerations. We relegate the details into Appendix C and quote the final result,

$$\mathcal{R}_c(k) = \delta N = -\frac{1}{2 f'(n_p)} \frac{(\delta\psi^2)_k}{\langle \delta\psi^2(0) \rangle} + \left[1 - \frac{f'(n_k)}{f'(n_p)} \right] \left(\frac{\delta\phi}{r\phi} \right)_k; \quad k > k_c. \quad (6.9)$$

As is seen from this result, the curvature perturbation spectrum is smoothly taken over from $\delta\phi$ to $\delta\psi^2$ as n increases from 0 to n_p .

As mentioned before, during the second stage the power spectrum can run significantly from blue to red. To avoid this one has to make sure the phase transition happens late enough during inflation so the contributions of these modes are negligible on large scale power spectrum. Following the analysis in [10] (for a method different than δN see also [8]) the contribution of the waterfall field in total curvature perturbation is ¹

$$\mathcal{P}_{\mathcal{R}}^{(\delta\psi)} \simeq \frac{\kappa_\psi^{-11/3}}{n_p} \left(\frac{k}{k_c} \right)^3. \quad (6.10)$$

The cosmological large scales relevant for CMB exit the horizon about $60 - 53$ e -folds before the end of inflation, so one should check that $\mathcal{P}_{\mathcal{R}}^{(\delta\psi)}$ is smaller than the observed COBE normalization $\mathcal{P}_{\mathcal{R}} \sim 10^{-10}$. This yields

$$\kappa_\psi^{-11/3} e^{-3(N_c-7)} \ll 10^{-10}, \quad (6.11)$$

which leads to the following constraint

$$N_c > 14.7 - \frac{11}{9} \ln \kappa_\psi. \quad (6.12)$$

The above relation indicates that if one takes κ_ψ large enough then $\mathcal{P}_{\mathcal{R}}^{(\delta\psi)}$ is so suppressed that the transition can take place for $N_c \gtrsim 1$. Below we study each stage in brief.

B. $\phi_c^2 < 12M_P^2$

As discussed in section II inflation has four stages $N_{tot} = N_c + n_{trig} + \Delta n_{\phi-trans} + \Delta n_{\psi-trans}$ in which the ϕ transition period is very short.

Stage.1 This stage, corresponding to $N < N_c$, is completely similar to the case studied before.

Stage.2 This stage corresponds to the case where the back-reaction of ψ on ϕ becomes important triggering it to move towards its global minimum $\phi = 0$. The value of the ψ at the time of n_{trig} has no dependence on inflaton field so fluctuations of the inflaton field does not affect the duration of this period. Furthermore, one should now replace n_p in previous subsection by n_{trig} because the transition of ψ shuts off at n_{trig} instead of n_p . One can show that

$$n_{trig}^2 = \frac{1}{2\kappa_\psi} \ln \left(\frac{\psi_{min}}{\psi(0)} \frac{\phi_c}{M_P} \right). \quad (6.13)$$

¹ Note that the notation α used in [10] can be translated to our κ_ψ here as $\alpha = \sqrt{6\kappa_\psi}$

As in the previous sub-section suppose after the phase transition $\psi \sim \psi_c e^{f(n)}$ where the form of $f(n)$ depends on the sharpness of the phase transition. Following the same steps as in Section IV C, the contributions of ψ fluctuations into the variation of the waterfall transition number of e -folds is

$$\delta n_{trig} = \frac{-1}{2f'(n_{tri})} \ln \left(\frac{\delta\psi^2(0, \mathbf{x})}{\langle \delta\psi^2(0) \rangle} \right). \quad (6.14)$$

Regardless of the details of the transition, one deduces that the contribution of the waterfall field quantum fluctuations into the total curvature perturbations is

$$\mathcal{R}_c(\delta\phi = 0) = \delta n_{trig} = \frac{-1}{2f'(n_{tri})} \frac{(\delta\psi^2)_k}{\langle \delta\psi^2(0) \rangle}. \quad (6.15)$$

Stage.3 This stage corresponds to the short interval $\Delta n_{\phi-trans}$ when ϕ rolls quickly to its minimum. One can still consider the vacuum domination in this period as the waterfall field is far from its global minimum. The equation of the inflaton field can be read as follows

$$\phi'' + 3\phi' + \omega^2(n)\phi = 0 \quad (6.16)$$

in which

$$\omega^2(n) = \alpha + \frac{g^2}{H^2} \psi^2 \quad (6.17)$$

and $\psi \sim \psi_{trig} e^{f(n)}$. Defining $\phi = e^{-3/2n} \Phi$ one has

$$\Phi'' + \left(\frac{-9}{4} + \omega^2(n) \right) \Phi = 0. \quad (6.18)$$

Shortly after the triggering time the term containing ω^2 becomes large and completely dominates over $-9/4$ so one can use the WKB approximation to find the time dependence of the inflaton field during the fast roll towards $\phi = 0$,

$$\phi \sim e^{-3n/2} e^{\pm i \int^n \omega(n') dn'}. \quad (6.19)$$

This indicates ϕ scales like $e^{-3n/2}$ during its fast roll towards its minimum. The transition of ϕ , triggered due to back-reaction of the waterfall field, can be considered to be complete when the interaction term $g^2 \phi^2 \psi^2$ becomes negligible compared to the bare mass of the waterfall field

$$\phi^2 \ll \phi_c^2. \quad (6.20)$$

Since the inflaton field slow rolls before the triggering point, its amplitude there is about ϕ_c . One can readily concludes that the fast roll of inflaton field completes in

$$\Delta n_{\phi-trans} \sim 1. \quad (6.21)$$

So one can consider that this transition is somewhat instantaneous.

Stage.4 Soon after ϕ settles to its global minimum $\phi = 0$ the waterfall field experiences the large tachyonic mass $-M^2$.

One can check that in this period ψ starts growing like $\psi(n) \sim e^{\sqrt{\beta}n}$ towards its global minimum.

Again, to find the total curvature perturbation one should add up different contributions to δN from different periods specified above. If $\kappa_\psi \gg 1$, corresponding to a sharp phaser transition, the situation is the same as in standard hybrid inflation. Since the inflaton field is sub-Planckian then inflation can not continue after the waterfall in the form of chaotic inflation so one basically requires $N_c \simeq N_t \sim 60$. Furthermore, the large scale curvature perturbations do not get contribution from the waterfall field fluctuations which are suppressed by $(\frac{k}{k_c})^3 \sim e^{-3N_c} \ll 1$. Note that with the assumption $\kappa_\psi \gg 1$ we have $\beta \gg N_{total}$ i.e. $\beta \sim 10^2 - 10^3$.

However, if we assume $\kappa_\psi \ll 1$ so the phase transition is mild, then we face the same problem as discussed in section V, that is the curvature perturbations cannot be normalized to the COBE normalization. This can be seen from Eqs. (6.14) which has the same structure as Eqs. (4.24) and (4.25).

VII. SUMMARY AND DISCUSSIONS

In this paper we have looked into the parameter space of hybrid inflation. We have retained two key properties of the standard hybrid inflation scenario, the vacuum domination condition and the condition that the waterfall field is very heavy during early stage of inflation. The latter condition is equivalent to $\beta \gg 1$ causing the waterfall field to quickly rolls to its instantaneous minimum $\psi = 0$. Besides these two conditions, we have relaxed other conditions on the hybrid inflation parameters and searched for the limits of the parameter space where one can obtain curvature perturbations with the right amplitude. As we have seen, two key parameters in our classification are ϕ_c/M_P and κ_ψ where the latter parametrizes the sharpness of the phase transition. For a sharp phase transition, as in standard hybrid inflation, one has $\kappa_\psi \gg 1$, whereas for a mild phase transition $\kappa_\psi \ll 1$.

We have shown that models with $\phi_c^2 > 12M_P^2$ and $\kappa_\psi \ll 1$ do not work because they can not satisfy the observed COBE normalization. Note that to have $\kappa_\psi \ll 1$ while keeping the waterfall field heavy requires that $\beta \sim 10$ or so. In this model inflation has three extended stages, the first stage is before the waterfall, the second stage starts when the waterfall rolls mildly to its minimum while inflaton is slowly rolling. The final stage is similar to conventional chaotic inflation models. As we have seen, the dominant contributions into the curvature perturbation power spectrum come from the waterfall fluctuations. On the other hand, there is no background classical ψ trajectory so one identifies $\sqrt{\langle \delta\psi^2(n) \rangle}$ as the effective background trajectory. As a result, one finds that there is not a very small free parameter to tune the ratio $(k^3/(2\pi)^3)|\delta\psi_k(0)|^2/\langle \delta\psi^2(0) \rangle \sim \sqrt{\kappa_\psi}$ to the small observed value. This is in contrast with, say the $m^2\phi^2$ chaotic model, in which $\delta\phi \sim H$ while ϕ is kept as an initial value free parameter so one can tune $\delta\phi/\phi$ into an arbitrary small number. On the other hand models with $\phi_c^2 > 12M_P^2$ and $\kappa_\psi \gg 1$ can be allowed. The contributions of waterfall field fluctuations are the same as in standard hybrid inflation. However, one has to satisfy inequality (6.12) in order to avoid a significant running of spectrum from blue to red on the CMB scales.

Models with $\phi_c^2 < 12M_P^2$ and $\kappa_\psi \gg 1$ are similar to standard hybrid inflation and one can satisfy the COBE/WMAP normalization. In these models inflation end abruptly after the waterfall phase transition. For having $\kappa_\psi \gg 1$, one requires that $\beta \sim 10^2 - 10^3$. However, in models with $\phi_c^2 < 12M_P^2$ and $\kappa_\psi \ll 1$ one encounters the same problem as above that the COBE/WMAP normalization cannot be satisfied. This corresponds to $\beta \sim 10$ or so.

After these discussions, one concludes that models of hybrid inflation with the assumptions of vacuum domination and a mild phase transition do not work because one can not normalize the curvature perturbation power spectrum to the required COBE/WMAP normalization. This corresponds to models with $\kappa_\psi \ll 1$ and $\beta \sim 10$ or so.

Acknowledgement

We would like to thank David Lyth, Mohammad Hossein Namjoo and David Wands for useful discussions. A.A.A. would like to thank YITP for the hospitality where this work started. H.F. would like to thank ICG for the hospitality where this work was completed. M.S. would like to thank KIAS for the hospitality where the final part of this work was done. This work was supported in part by MEXT Grant-in-Aid for the global COE program at Kyoto University, "The Next Generation of Physics, Spun from Universality and Emergence," and by Korea Institute for Advanced Study under the KIAS Scholar program. This work was also supported in part by JSPS Grant-in-Aid for Scientific Research (A) No. 21244033, and by JSPS Grant-in-Aid for Creative Scientific Research No. 19GS0219.

Appendix A: Justifying integrating out sub-horizon modes in Eq. (4.10)

Here we justify our approximation in integrating the amplitudes of all of quantum modes regardless of being super-horizon or sub-horizon at critical point, given by Eq. (4.10). In order to employ the δN formalism we have to start with a well-defined background classical trajectory [26]. In our model, there is no well-defined background trajectory for the ψ field. However, the accumulative effects of quantum modes $\delta\psi$ defines the effective ψ background as $\sqrt{\langle \delta\psi^2 \rangle}$. To find this average we should consider the modes which are at any time super-horizon. In other words, we renormalize the two point function with cut-off defined by the comoving momentum $\Lambda = k/a = H$. By the above discussion the background ψ_b at any time can be read as follows

$$\psi_b^2 = \langle \delta\psi^2 \rangle_{ren.}(n) = \int_0^{k=aH_0} \frac{d^3k}{(2\pi)^3} |\delta\psi_k^{(0)}|^2 e^{2\kappa_\psi n^2}. \quad (A1)$$

By using Eq. (4.9) one can simplify the above integral

$$\psi_b^2 = \frac{H_0^2}{4\pi^2} \int_{-\infty}^n dn_k e^{-2\kappa_\psi n_k^2} e^{2\kappa_\psi n^2} \quad (A2)$$

In the above integral, at any given time, only the super-horizon modes are considered. The above integral can be found in terms of error-functions

$$\psi_b^2 = \frac{H_0^2}{8\pi^2} \sqrt{\frac{\pi}{2\kappa_\psi}} (1 + \text{Erf}(\sqrt{2\kappa_\psi}n)) e^{2\kappa_\psi n^2}. \quad (\text{A3})$$

By using the properties of the error-functions, one can verify that for the time $n > 1/\sqrt{2\kappa_\psi}$ the above expression approaches quickly to

$$\psi_b^2 \simeq \frac{H_0^2}{4\pi^2} \sqrt{\frac{\pi}{2\kappa_\psi}} e^{2\kappa_\psi n^2} = \langle \delta\psi^2(0) \rangle e^{2\kappa_\psi n^2} \quad (n > 1/\sqrt{2\kappa_\psi}). \quad (\text{A4})$$

Now let us compare $1/\sqrt{\kappa_\psi}$ with n_p : the bigger is the value of $\sqrt{\kappa_\psi}n_p > 1$ the better gets our approximation Eq. (A4). Let us now have an estimate for $\sqrt{\kappa_\psi}n_p$. From Eq. (4.17) we have

$$\kappa_\psi n_p^2 \simeq -\frac{1}{2} \ln \left(\frac{\beta}{\lambda} \sqrt{32\pi^3 \kappa_\psi} \right). \quad (\text{A5})$$

The term inside the square root is basically at the order of unity, therefore $\kappa_\psi n_p^2 \simeq -\ln \sqrt{\lambda}$. For λ sufficiently small one can arrange for $\kappa_\psi n_p^2 \gg 1$. For example in our numerical analysis, $\lambda \sim 10^{-9}$ which gives $\kappa_\psi n_p^2 \simeq 21$ so our approximation Eq. (A4) is very good. Therefore for times well after the phase transition but before the phase transition completion one can use the approximation (A4) as a good estimate for the background ψ field.

In the δN formalism the evolution of curvature perturbations is considered only for super-horizon modes [25, 26]. Working on super-horizon scales means smoothing out the sub-horizon perturbations in a patch. At any time super-horizon modes are the modes with the comoving wavelength larger than the Hubble radius, $k < aH$. Therefore one can integrate the curvature perturbations by considering only the super-horizon modes at a given time. For example in our model, at the time of phase transition there are many modes which are sub-horizon, but as time goes by these modes become super-horizon and should be considered in the background evolution. This point is what we considered in evaluating the average trajectory of ψ field. Therefore to calculate $\delta\psi^2(x)$, defined on every patch, only the super-horizon modes should be taken into account. This qualitative discussion verifies that for each patch and for times well after the critical point one can use the approximation

$$\delta\psi^2(x, n) \simeq \delta\psi^2(x, 0) e^{2\kappa_\psi n^2}, \quad n \gtrsim \frac{1}{\sqrt{2\kappa_\psi}}. \quad (\text{A6})$$

The above arguments support our discussions at the main text about the main n -dependence of $\delta\psi^2$ for every patch.

Appendix B: Careful treatments of the waterfall induced curvature perturbations

Here we study in details the integral in Eq. (5.3) to find the induced curvature perturbations from the waterfall field.

From Eq. (4.9) for the mode function $\delta\psi_p$, we have

$$|\delta\psi_{|\mathbf{k}-\mathbf{q}|}(0)|^2 = \frac{H^2}{2k_c^3} \exp \left[-3n_{|\mathbf{k}-\mathbf{q}|} - 2\kappa_\psi n_{|\mathbf{k}-\mathbf{q}|}^2 \right], \quad (\text{B1})$$

where

$$n_{|\mathbf{k}-\mathbf{q}|} = \frac{1}{2} \ln \left[\frac{k^2 + q^2 - 2qk \cos \theta}{k_c^2} \right], \quad (\text{B2})$$

and $\kappa_\psi = \beta r/3$.

The momentum integral contains an angular part plus radial part. Taking account of the azimuthal symmetry, we can write

$$\int d^3q = \int 2\pi q^2 dq \int d(\cos \theta) \quad (\text{B3})$$

where $\cos \theta = \mathbf{k} \cdot \mathbf{q}/kq$. Changing the integration variable from $\cos \theta$ to $n_{|\mathbf{k}-\mathbf{q}|}$, the angular integral becomes

$$\int d(\cos \theta) |\delta\psi_{|\mathbf{k}-\mathbf{q}|}(0)|^2 = \frac{H^2}{2k_c q k} \int_{n_{|\mathbf{k}-\mathbf{q}|}}^{n_{k+q}} dn \exp[-n - 2\kappa_\psi n^2]. \quad (\text{B4})$$

The result of the above integral is in the form of the error functions,

$$\int d(\cos \theta) |\delta\psi_{|\mathbf{k}-\mathbf{q}|}(0)|^2 = \frac{H^2}{2k_c q k} \sqrt{\frac{\pi}{2\kappa_\psi}} e^{1/(8\kappa_\psi)} \left[\text{Erf} \left(\frac{1 + 4\kappa_\psi n_{k+q}}{2\sqrt{2\kappa_\psi}} \right) - \text{Erf} \left(\frac{1 + 4\kappa_\psi n_{|\mathbf{k}-\mathbf{q}|}}{2\sqrt{2\kappa_\psi}} \right) \right], \quad (\text{B5})$$

where

$$n_{k+q} = \ln \left(\frac{k+q}{k_c} \right), \quad (\text{B6})$$

$$n_{|\mathbf{k}-\mathbf{q}|} = \ln \left(\frac{|k-q|}{k_c} \right), \quad (\text{B7})$$

and the error function is defined by

$$\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \quad (\text{B8})$$

Plugging Eq. (B5) into the total momentum integral one has

$$\begin{aligned} \langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \rangle &= \delta^3(\mathbf{k} + \mathbf{k}') \times \\ &2\pi H^2 \sqrt{\frac{\pi}{2\kappa_\psi}} e^{1/(8\kappa_\psi)} \int \frac{qdq}{k_c k} |\delta\psi_q(0)|^2 \left[\text{Erf} \left(\frac{1 + 4\kappa_\psi n_{k+q}}{2\sqrt{2\kappa_\psi}} \right) - \text{Erf} \left(\frac{1 + 4\kappa_\psi n_{|\mathbf{k}-\mathbf{q}|}}{2\sqrt{2\kappa_\psi}} \right) \right]. \end{aligned} \quad (\text{B9})$$

The error function, $\text{Erf}(z)$ varies from -1 for $z \rightarrow -\infty$ to $+1$ for $z \rightarrow \infty$. This transition is sharp and take place near $z = 0$. One can see that the overall behavior of this function is similar to $\tanh(z)$. For k not extremely small, the argument of the first error function in the square brackets is always much greater than unity, hence this term is almost 1. The second term depends on the difference between the amplitudes of two momenta, $|k-q|$, and can change from -1 for $q = k$ to 1 for q completely different from k . There is two criteria, first, a very narrow region in which the argument of the second error function become small or negative, in this region the difference between two error functions is considerable but the domain of the integration is very small. Second possibility is when the amplitude of the integrand is not too large but the domain of integration is large. This point is more clarified in the following

We showed in the draft that the contribution of the modes in the first criteria is suppressed by a factor $e^{-1/8\kappa_\psi}$ as follows on the first contribution to the integral.

$$\left\langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \right\rangle_1 = 2(2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{H^2}{4\pi^2} \sqrt{\frac{\pi}{2\kappa_\psi}} e^{-1/(8\kappa_\psi)} |\delta\psi_{\mathbf{k}}(0)|^2 \quad (\text{B10})$$

It's clear from the steps of the calculation that this suppression is due to narrow width of the integration domain. It's time to calculate the second contribution. This contribution comes from the region in which the arguments of two error functions in the big bracket are not too far apart and hence the difference between the error functions is small. As we pointed out before, the argument of the first error function is large and it's amplitude of is approximately equal to unity. outside the first regime discussed above the amplitude of the second error function is also large so we can use the following estimate for the behavior of the error functions with large argument

$$(1 - \text{Erf}z) \sim \frac{e^{-z^2}}{\sqrt{\pi}z} \quad (\text{B11})$$

Let us first clarify a tiny point. We defined before the first contribution to the integral comes from the region in which

$$\frac{1 + 4\kappa_\psi n_{|\mathbf{k}-\mathbf{q}|}}{2\sqrt{2\kappa_\psi}} < z_{bw} \sim 1 \quad (\text{B12})$$

and here we are going to concentrate on the region in which

$$\frac{1 + 4\kappa_\psi n_{|\mathbf{k}-\mathbf{q}|}}{2\sqrt{2\kappa_\psi}} > 1 \quad (\text{B13})$$

one can say as $z_{bw} \simeq 1/4$ you may miss some of contribution to the integral but we claim that this error is not considerable. By using the Eq. (B11) one has

$$\text{Erf} \left(\frac{1 + 4\kappa_\psi n_{k+q}}{2\sqrt{2\kappa_\psi}} \right) - \text{Erf} \left(\frac{1 + 4\kappa_\psi n_{|k-q|}}{2\sqrt{2\kappa_\psi}} \right) \simeq 2\sqrt{\frac{2\kappa_\psi}{\pi}} \frac{e^{-1/8\kappa_\psi}}{1 + 4\kappa_\psi n_{|k-q|}} \exp \left(-n_{|k-q|} - 2\kappa_\psi n_{|k-q|}^2 \right)$$

One can also more simplify the above equation by ignoring the term $+4\kappa_\psi n_{|k-q|}$ in the denominator of the first fraction. This approximation can be verified by noting that the total term has as $\exp \left(2\kappa_\psi n_{|k-q|}^2 \right)$ dependence.

By using the above approximation one has for the second contribution to the integral of the two point correlation function

$$\langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \rangle_2 = \delta^3(\mathbf{k} + \mathbf{k}') \times 4\pi H^2 \int \frac{q dq}{k_c k} |\delta\psi_q(0)|^2 \exp \left(-n_{|k-q|} - 2\kappa_\psi n_{|k-q|}^2 \right). \quad (\text{B14})$$

By noting that $n_{|k-q|} = \ln \frac{|k-q|}{k_c}$, one has

$$dq = \pm k_c e^{n_{|k-q|}} dn_{|k-q|} \quad (\text{B15})$$

respectively for $q > k$ and $q < k$. Again we claim that as this combination of the error functions are highly peaked around the $q = k$ we take the explicit q -dependences out of the integral and identify q in them by k . By this trick and noting that the remaining integrand is even one has

$$\langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \rangle_2 \simeq \delta^3(\mathbf{k} + \mathbf{k}') \times 8\pi H^2 |\delta\psi_k(0)|^2 \int dn_{|k-q|} \exp \left(-2\kappa_\psi n_{|k-q|}^2 \right). \quad (\text{B16})$$

By performing the integration finally we have

$$\langle (\delta\psi^2)_{\mathbf{k}} (\delta\psi^2)_{\mathbf{k}'} \rangle_2 \simeq (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \times \frac{1}{2} \langle \delta\psi^2(0) \rangle |\delta\psi_k(0)|^2 \quad (\text{B17})$$

This relation is in complete agreement with our previous estimate for the two point correlation function, Eq. (5.5)

Appendix C: Curvature perturbations for $k > k_c$ with $\phi_c^2 > 12M_p^2, \kappa_\psi \gg 1$

In this appendix we are going to study the contribution of the quantum fluctuations with $k > k_c$ to the final curvature perturbations for the case $\phi_c^2 > 12M_p^2$, using the δN formalism. For this purpose, we trace back the number of e -folds, \mathcal{N} , from the end of inflation until time of horizon crossing of some specific mode during transition n . To avoid confusion, in this appendix we denote the number of e -folds counted backward from end of inflation by \mathcal{N} . Namely, $\mathcal{N} = n - n_f$ where n_f is the number of e -folds from the critical point $\phi = \phi_c$ until the end of inflation. Our strategy is to express \mathcal{N} in terms of the fields $\phi(n)$ and $\psi^2(n)$ (smoothed on every Hubble patch).

As discussed in sections II and III, in the models with $\phi_c^2 > 12M_p^2$, the transition is followed by a chaotic period of inflation. So the end of inflation is determined by the value of ϕ alone as

$$\phi = \phi_f \approx \sqrt{2/3} M_P. \quad (\text{C1})$$

From this up to the end of the waterfall transition \mathcal{N} is given by

$$\mathcal{N} = \frac{\phi^2 - \phi_f^2}{4M_P^2} \approx \frac{\phi^2}{4M_P^2}; \quad \mathcal{N} \leq \mathcal{N}_p, \quad (\text{C2})$$

where \mathcal{N}_p is the value of \mathcal{N} at the end of waterfall transition. As shown in Eq. (3.12) the value of ψ^2 at $\mathcal{N} = \mathcal{N}_p$ is

$$\psi^2(n_p) = \psi_{min}^2(n_p) = \frac{M^2 - g^2 \phi(\mathcal{N}_p)^2}{\lambda} = \frac{M^2 - 4g^2 \mathcal{N}_p M_P^2}{\lambda}. \quad (\text{C3})$$

Here \mathcal{N}_p is given in terms of the number of e -folds from the critical epoch $\phi = \phi_c$ to the end of waterfall transition, which is denoted by n_p , as

$$\mathcal{N}_p(n_p) = \frac{\phi_p^2}{4M_P^2} = \frac{\phi_c^2 e^{-2rn_p}}{4M_P^2}. \quad (\text{C4})$$

Hence $\psi^2(\mathcal{N}_p)$ is a function of n_p through its dependence on $\phi(\mathcal{N}_p)$,

$$\psi^2(\mathcal{N}_p) = \psi_{min}^2(n_p) = \frac{M^2}{\lambda} - 4\frac{g^2 M_P^2}{\lambda} \mathcal{N}_p(n_p). \quad (\text{C5})$$

Now we trace back the evolution to earlier times before the end of transition, $\mathcal{N} > \mathcal{N}_p$. For this stage, instead of \mathcal{N} , it is more convenient to use n which is the number of e -folds from the critical point counted *forward* in time, ie, $n = n_p + \mathcal{N}_p - \mathcal{N}$. Then $\psi^2(n)$ is given by

$$\psi^2(n) = \exp 2[f(n) - f(n_p)] \psi_{min}^2(n_p), \quad (\text{C6})$$

where $f(n)$ a function that describes the tachyonic growth of the waterfall field, as given by Eq. (6.5). During this era, as for ϕ , we have

$$\phi(n) = \phi_c e^{-rn}. \quad (\text{C7})$$

Here we note that n depends on n_p and \mathcal{N} in a rather non-trivial way,

$$n(n_p, \mathcal{N}) = n_p + \mathcal{N}_p(n_p) - \mathcal{N}. \quad (\text{C8})$$

In particular, we have

$$\frac{\partial n}{\partial n_p} = 1 + \frac{d\mathcal{N}_p}{dn_p} = 1 - 2r\mathcal{N}_p, \quad (\text{C9})$$

where we have used the fact that

$$\frac{d\mathcal{N}_p}{dn_p} = -2r\mathcal{N}_p, \quad (\text{C10})$$

which follows from Eq. (C4).

Keeping in mind the above dependence of n on n_p and \mathcal{N} , let us take the variation of (C6) and (C7). We obtain

$$\begin{aligned} \delta\psi^2(n) &= -2\psi^2(n)f'(n)\delta\mathcal{N} \\ &\quad + \psi^2(n) \left(2 \left[f'(n) \frac{\partial n}{\partial n_p} - f'(n_p) \right] + \frac{d \ln \psi_{min}^2(n_p)}{dn_p} \right) \delta n_p. \end{aligned} \quad (\text{C11})$$

$$\delta\phi(n) = -r\phi(n)\delta n = r\phi(n)\delta\mathcal{N} - r\phi(n) \frac{\partial n}{\partial n_p} \delta n_p. \quad (\text{C12})$$

By using Eq. (C5), one finds

$$\left| \frac{d \ln \psi_{min}^2(n_p)}{dn_p} \right| \simeq 8g^2 \frac{M_P^2}{M^2} r\mathcal{N}_p \lesssim 2r \ll 1, \quad (\text{C13})$$

where we have used Eq. (C2) and the fact that $\phi_p \lesssim \phi_c$ in the last step.

To find the variations of the number of e -folds, one solves Eqs. (C11) and (C12) for $\delta\mathcal{N}$ to obtain,

$$\delta\mathcal{N} = \frac{\delta\psi^2(n)}{\psi^2(n)} \frac{\partial n}{\partial n_p} \frac{1}{-2f'(n_p)} + \frac{\delta\phi(n)}{r\phi(n)} \left[1 + \frac{2f'(n)}{-2f'(n_p)} \frac{\partial n}{\partial n_p} \right], \quad (\text{C14})$$

in which we have ignored the term $|d \ln \psi_p^2(n_p)/dn_p| \sim r \ll 1$. Note that as a result of the vacuum domination condition, we also have $|d\mathcal{N}_p/dn_p| = 2r\mathcal{N}_p \ll 1$. Then the above reduces to

$$\delta\mathcal{N} = -\frac{1}{2f'(n_p)} \frac{\delta\psi^2(n)}{\psi^2(n)} + \left[1 - \frac{f'(n)}{f'(n_p)} \right] \frac{\delta\phi(n)}{r\phi(n)}. \quad (\text{C15})$$

As clear from this result, the curvature perturbation spectrum is smoothly taken over from $\delta\phi$ to $\delta\psi^2$ as n increases from 0 to n_p .

References

-
- [1] E. Komatsu *et al.*, arXiv:1001.4538 [astro-ph.CO].
 - [2] A. H. Guth, Phys. Rev. D **23**, 347 (1981);
K. Sato, Mon. Not. Roy. Astron. Soc. **195**, 467-479 (1981).
 - [3] A. D. Linde, Phys. Lett. B **108**, 389 (1982);
A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
 - [4] A. D. Linde, Phys. Lett. **B129**, 177-181 (1983).
 - [5] A. D. Linde, Phys. Rev. D **49**, 748 (1994) [arXiv:astro-ph/9307002].
 - [6] E. J. Copeland *et al.*, Phys. Rev. D **49**, 6410 (1994) [arXiv:astro-ph/9401011].
 - [7] D. H. Lyth, arXiv:1005.2461 [astro-ph.CO].
 - [8] A. A. Abolhasani, H. Firouzjahi, Phys. Rev. **D83**, 063513 (2011). [arXiv:1005.2934 [hep-th]].
 - [9] J. Fonseca, M. Sasaki, D. Wands, JCAP **1009**, 012 (2010). [arXiv:1005.4053 [astro-ph.CO]].
 - [10] J. -O. Gong, M. Sasaki, JCAP **1103**, 028 (2011). [arXiv:1010.3405 [astro-ph.CO]].
 - [11] D. H. Lyth, [arXiv:1012.4617 [astro-ph.CO]].
 - [12] L. P. Levasseur, G. Laporte, R. Brandenberger, Phys. Rev. **D82**, 123524 (2010). [arXiv:1004.1425 [hep-th]].
 - [13] N. Barnaby and J. M. Cline, Phys. Rev. D **75**, 086004 (2007) [arXiv:astro-ph/0611750];
N. Barnaby and J. M. Cline, Phys. Rev. D **73**, 106012 (2006) [arXiv:astro-ph/0601481].
 - [14] A. Mazumdar, J. Rocher, Phys. Rept. **497**, 85-215 (2011). [arXiv:1001.0993 [hep-ph]].
 - [15] K. Enqvist, A. Jokinen, A. Mazumdar, T. Multamaki and A. Vaihkonen, Phys. Rev. Lett. **94**, 161301 (2005) [arXiv:astro-ph/0411394];
K. Enqvist, A. Jokinen, A. Mazumdar, T. Multamaki and A. Vaihkonen, JCAP **0503**, 010 (2005) [arXiv:hep-ph/0501076].
 - [16] K. Enqvist and A. Vaihkonen, JCAP **0409**, 006 (2004) [arXiv:hep-ph/0405103].
 - [17] L. Randall, M. Soljagic, A. H. Guth, Nucl. Phys. **B472**, 377-408 (1996). [hep-ph/9512439].
 - [18] J. Garcia-Bellido, A. D. Linde, D. Wands, Phys. Rev. **D54**, 6040-6058 (1996). [astro-ph/9605094].
 - [19] A. A. Abolhasani, H. Firouzjahi, M. H. Namjoo, [arXiv:1010.6292 [astro-ph.CO]].
 - [20] J. Silk and M. S. Turner, Phys. Rev. D **35**, 419 (1987).
D. Polarski and A. A. Starobinsky, Nucl. Phys. B **385**, 623 (1992);
D. Polarski and A. A. Starobinsky, Phys. Rev. D **50**, 6123 (1994) [arXiv:astro-ph/9404061];
D. Langlois, Phys. Rev. D **59**, 123512 (1999) [arXiv:astro-ph/9906080]. S. Tsujikawa, D. Parkinson and B. A. Bassett,
Phys. Rev. D **67**, 083516 (2003) [arXiv:astro-ph/0210322];
 - [21] C. L. Bennett *et al.*, Astrophys. J. **464**, L1 (1996) [arXiv:astro-ph/9601067].
 - [22] A. R. Liddle, D. Parkinson, S. M. Leach and P. Mukherjee, Phys. Rev. D **74**, 083512 (2006) [arXiv:astro-ph/0607275].
 - [23] S. Clesse, Phys. Rev. **D83**, 063518 (2011). [arXiv:1006.4522 [gr-qc]].
 - [24] H. Kodama, K. Kohri, K. Nakayama, [arXiv:1102.5612 [astro-ph.CO]].
 - [25] M. Sasaki, E. D. Stewart, Prog. Theor. Phys. **95**, 71-78 (1996). [astro-ph/9507001].
 - [26] D. Wands, K. A. Malik, D. H. Lyth *et al.*, Phys. Rev. **D62**, 043527 (2000). [astro-ph/0003278].